1. (a) We needn't consider pairs where the first number is larger than the second, since we can just interchange the numbers in such cases. The answer appears to be 11 and 12, but we have considered only integers in the table.

<table>
<thead>
<tr>
<th>First Number</th>
<th>Second Number</th>
<th>Product</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>22</td>
<td>22</td>
</tr>
<tr>
<td>2</td>
<td>21</td>
<td>42</td>
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<td>102</td>
</tr>
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<td>7</td>
<td>16</td>
<td>112</td>
</tr>
<tr>
<td>8</td>
<td>15</td>
<td>120</td>
</tr>
<tr>
<td>9</td>
<td>14</td>
<td>126</td>
</tr>
<tr>
<td>10</td>
<td>13</td>
<td>130</td>
</tr>
<tr>
<td>11</td>
<td>12</td>
<td>132</td>
</tr>
</tbody>
</table>

(b) Call the two numbers \( x \) and \( y \). Then \( x + y = 23 \), so \( y = 23 - x \). Call the product \( P \). Then

\[
P = xy = x(23 - x) = 23x - x^2,
\]

so we wish to maximize the function

\[
P(x) = 23x - x^2.
\]

Since \( P'(x) = 23 - 2x \), we see that \( P'(x) = 0 \) \( \iff \) \( x = \frac{23}{2} = 11.5 \). Thus, the maximum value of \( P \) is \( P(11.5) = (11.5)^2 = 132.25 \) and it occurs when \( x = y = 11.5 \).

Or: Note that \( P''(x) = -2 < 0 \) for all \( x \), so \( P \) is everywhere concave downward and the local maximum at \( x = 11.5 \) must be an absolute maximum.

3. The two numbers are \( x \) and \( \frac{100}{x} \), where \( x > 0 \). Minimize \( f(x) = x + \frac{100}{x} \). \( f'(x) = 1 - \frac{100}{x^2} = \frac{x^2 - 100}{x^2} \). The critical number is \( x = 10 \). Since \( f'(x) < 0 \) for \( 0 < x < 10 \) and \( f'(x) > 0 \) for \( x > 10 \), there is an absolute minimum at \( x = 10 \). The numbers are 10 and 10.

5. If the rectangle has dimensions \( x \) and \( y \), then its perimeter is \( 2x + 2y = 100 \) m, so \( y = 50 - x \). Thus, the area is

\[
A = xy = x(50 - x).
\]

We wish to maximize the function \( A(x) = x(50 - x) = 50x - x^2 \), where \( 0 < x < 50 \). Since \( A'(x) = 50 - 2x = -2(x - 25), A'(x) > 0 \) for \( 0 < x < 25 \) and \( A'(x) < 0 \) for \( 25 < x < 50 \). Thus, \( A \) has an absolute maximum at \( x = 25 \), and \( A(25) = 25^2 = 625 \) m². The dimensions of the rectangle that maximize its area are \( x = y = 25 \) m. (The rectangle is a square.)
10. (a) 

The volumes of the resulting boxes are 1, 1.6875, and 2 ft³. There appears to be a maximum volume of at least 2 ft³.

(b) Let \( x \) denote the length of the side of the square being cut out. Let \( y \) denote the length of the base.

c) Volume \( V = \text{length} \times \text{width} \times \text{height} \Rightarrow V = y \cdot y \cdot x = xy^2 \)

(d) Length of cardboard = 3 \( \Rightarrow x + y + x = 3 \Rightarrow y + 2x = 3 \)

(e) \( y + 2x = 3 \Rightarrow y = 3 - 2x \Rightarrow V(x) = x(3 - 2x)^2 \)

(f) \( V(x) = x(3 - 2x)^2 \Rightarrow V'(x) = 2(3 - 2x)(-2) + (3 - 2x)^2 \cdot 1 = (3 - 2x)[-4x + (3 - 2x)] = (3 - 2x)(-6x + 3), \)

so the critical numbers are \( x = \frac{2}{3} \) and \( x = \frac{1}{2} \). Now \( 0 \leq x \leq \frac{3}{2} \) and \( V(0) = V\left(\frac{3}{2}\right) = 0 \), so the maximum is \( V\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^2(2) = 2 \) ft³, which is the value found from our third figure in part (a).

13. Let \( b \) be the length of the base of the box and \( h \) the height. The surface area is \( 1200 = b^2 + 4bh \Rightarrow h = \frac{(1200 - b^2)}{(4b)}. \)

The volume is \( V = b^2h = b^2\left(1200 - b^2\right)/4b = \frac{300b - b^3}{4} \Rightarrow V'(b) = 300 - \frac{3}{4}b^2 \).

\( V'(b) = 0 \Rightarrow 300 = \frac{3}{4}b^2 \Rightarrow b^2 = 400 \Rightarrow b = \sqrt{400} = 20. \) Since \( V'(b) > 0 \) for \( 0 < b < 20 \) and \( V'(b) < 0 \) for \( b > 20 \), there is an absolute maximum when \( b = 20 \) by the First Derivative Test for Absolute Extreme Values (see page 324).

If \( b = 20 \), then \( h = \frac{(1200 - 20^2)}{(4 \cdot 20)} = 10 \), so the largest possible volume is \( b^2h = (20)^2(10) = 4000 \) cm³.

16. (a) Let the rectangle have sides \( x \) and \( y \) and area \( A \), so \( A = xy \) or \( y = A/x \). The problem is to minimize the perimeter = \( 2x + 2y = 2x + 2A/x = P(x) \). Now \( P'(x) = 2 - 2A/x^2 = 2(x^2 - A)/x^2 \). So the critical number is \( x = \sqrt{A} \). Since \( P'(x) < 0 \) for \( 0 < x < \sqrt{A} \) and \( P'(x) > 0 \) for \( x > \sqrt{A} \), there is an absolute minimum at \( x = \sqrt{A} \).

The sides of the rectangle are \( \sqrt{A} \) and \( A/\sqrt{A} = \sqrt{A} \), so the rectangle is a square.

(b) Let \( p \) be the perimeter and \( x \) and \( y \) the lengths of the sides, so \( p = 2x + 2y \Rightarrow 2y = p - 2x \Rightarrow y = \frac{1}{2}p - x \).

The area is \( A(x) = x\left(\frac{1}{2}p - x\right) = \frac{1}{2}px - x^2 \). Now \( A'(x) = 0 \Rightarrow \frac{1}{2}p - 2x = 0 \Rightarrow 2x = \frac{1}{2}p \Rightarrow x = \frac{1}{4}p \). Since \( A''(x) = -2 < 0 \), there is an absolute maximum for \( A \) when \( x = \frac{1}{4}p \) by the Second Derivative Test. The sides of the rectangle are \( \frac{1}{4}p \) and \( \frac{1}{2}p - \frac{1}{4}p = \frac{1}{4}p \), so the rectangle is a square.
17. The distance from a point \((x, y)\) on the line \(y = 4x + 7\) to the origin is \(\sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}\). However, it is easier to work with the square of the distance; that is, 
\[D(x) = \left(\sqrt{x^2 + y^2}\right)^2 = x^2 + y^2 = x^2 + (4x + 7)^2.\] Because the distance is positive, its minimum value will occur at the same point as the minimum value of \(D\).

\[D'(x) = 2x + 2(4x + 7)(4) = 34x + 56, \text{ so } D'(x) = 0 \iff x = -\frac{28}{17}.\]

\[D''(x) = 34 > 0, \text{ so } D \text{ is concave upward for all } x.\] Thus, \(D\) has an absolute minimum at \(x = -\frac{28}{17}\). The point closest to the origin is \((x, y) = \left(-\frac{28}{17}, 4\left(-\frac{28}{17}\right) + 7\right) = \left(-\frac{28}{17}, \frac{7}{17}\right)\).

22. The area of the rectangle is \((2x)(2y) = 4xy\). Now \(\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\) gives

\[y = \frac{b}{a}\sqrt{a^2 - x^2}, \text{ so we maximize } A(x) = 4\frac{b}{a}x\sqrt{a^2 - x^2}.\]

\[A'(x) = \frac{4b}{a} \left[ x \cdot \frac{1}{2} (a^2 - x^2)^{-1/2} (-2x) + (a^2 - x^2)^{1/2} \cdot 1 \right] = \frac{4b}{a} \frac{x^2 - x^2 - 1/2 [-x^3 + a^2 - x^2]}{a^2 - x^2} = \frac{4b}{a} \frac{x^3 + a^2 - x^2}{a^2 - x^2}\]

So the critical number is \(x = \frac{1}{\sqrt{2}} a\), and this clearly gives a maximum. Then \(y = \frac{1}{\sqrt{2}} b\), so the maximum area is \(4\left(\frac{1}{\sqrt{2}} a\right) \left(\frac{1}{\sqrt{2}} b\right) = 2ab\).

26. The rectangle has area \(xy\). By similar triangles \(\frac{3-y}{x} = \frac{3}{4} \iff -4y + 12 = 3x\) or \(y = -\frac{3}{4}x + 3\). So the area is

\[A(x) = x \left(-\frac{3}{4}x + 3\right) = -\frac{3}{4}x^2 + 3x \text{ where } 0 < x \leq 4.\] Now \(0 = A'(x) = -\frac{3}{2}x + 3 \iff x = 2\) and \(y = \frac{3}{2}\). Since \(A(0) = A(4) = 0\), the maximum area is \(A(2) = 2\left(\frac{3}{2}\right) = 3\text{ cm}^2\).

30. Perimeter = 30 \(\implies 2y + x + \pi\left(\frac{x}{2}\right) = 30 \implies y = \frac{1}{2}\left(30 - x - \frac{\pi x}{2}\right) = 15 - \frac{x}{2} - \frac{\pi x}{2}\). The area is the area of the rectangle plus the area of the semicircle, or \(xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2\), so \(A(x) = x\left(15 - \frac{x}{2} - \frac{\pi x}{4}\right) + \frac{1}{8}\pi x^2 = 15x - \frac{1}{4}x^2 - \frac{3}{8}\pi x^2\).

\[A'(x) = 15 - (1 + \frac{\pi}{4})x = 0 \implies x = \frac{60}{4 + \pi}. \quad A''(x) = -\left(1 + \frac{\pi}{4}\right) < 0, \text{ so this gives a maximum.}\]

The dimensions are \(x = \frac{60}{4 + \pi}\) ft and \(y = 15 - \frac{30}{4 + \pi} - \frac{15\pi}{4 + \pi} = \frac{60 + 15\pi - 30 - 15\pi}{4 + \pi} = \frac{30}{4 + \pi}\) ft, so the height of the rectangle is half the base.
33. Let \( \ell \) be the length of the wire used for the square. The total area is

\[
A(\ell) = \left( \frac{\ell}{4} \right)^2 + \frac{1}{2} \left( \frac{10 - \ell}{3} \right) \sqrt{\frac{3}{2}} \left( \frac{10 - \ell}{3} \right)
\]

\[
= \frac{1}{16} \ell^2 + \frac{\sqrt{3}}{18} (10 - \ell)^2, \quad 0 \leq \ell \leq 10
\]

\[A'(\ell) = \frac{1}{8} \ell - \frac{\sqrt{3}}{18} (10 - \ell) = 0 \quad \iff \quad \frac{9}{\sqrt{3}} \ell + \frac{3\sqrt{3}}{2} (10 - \ell) = 0 \quad \iff \quad \ell = \frac{49\sqrt{3}}{9 + 4\sqrt{3}} \approx 4.81, \]

\[A(10) = \frac{100}{18} = 6.25 \quad \text{and} \quad A\left(\frac{49\sqrt{3}}{9 + 4\sqrt{3}}\right) \approx 2.72, \quad \text{so}
\]

(a) The maximum area occurs when \( \ell = 10 \text{ m} \), and all the wire is used for the square.

(b) The minimum area occurs when \( \ell = \frac{49\sqrt{3}}{9 + 4\sqrt{3}} \approx 4.35 \text{ m} \).

50. The line with slope \( m \) (where \( m < 0 \)) through \((3, 5)\) has equation \( y - 5 = m(x - 3) \) or \( y = mx + (5 - 3m) \). The \( y \)-intercept is \( 5 - 3m \) and the \( x \)-intercept is \( -\frac{5}{m} + 3 \). So the triangle has area \( A(m) = \frac{1}{2}(5 - 3m)(-\frac{5}{m} + 3) = 15 - 25/(2m) - \frac{25}{2} \). Now

\[A'(m) = \frac{25}{2m^2} - \frac{9}{2} = 0 \quad \iff \quad m^2 = \frac{25}{9} \quad \Rightarrow \quad m = -\frac{5}{3} \quad \text{(since} \quad m < 0 \text{)}.
\]

\[A''(m) = \frac{-25}{m^3} > 0, \quad \text{so there is an absolute minimum when} \quad m = -\frac{5}{3} \quad \text{Thus, an equation of the line is} \quad y - 5 = -\frac{5}{3}(x - 3)
\]

or \( y = -\frac{5}{3}x + 10 \).

63. The total time is

\[T(x) = \text{time from } A \text{ to } C \text{ } + \text{ time from } C \text{ to } B
\]

\[= \frac{\sqrt{a^2 + x^2}}{v_1} + \frac{\sqrt{b^2 + (d-x)^2}}{v_2}, \quad 0 < x < d
\]

\[T'(x) = \frac{x}{v_1 \sqrt{a^2 + x^2}} - \frac{d-x}{v_2 \sqrt{b^2 + (d-x)^2}} = \frac{\sin \theta_1}{v_1} - \frac{\sin \theta_2}{v_2}
\]

The minimum occurs when \( T'(x) = 0 \quad \Rightarrow \quad \frac{\sin \theta_1}{v_1} = \frac{\sin \theta_2}{v_2} \).

[Note: \( T''(x) > 0 \)]
From the figure, \( \tan \alpha = \frac{5}{x} \) and \( \tan \beta = \frac{2}{3-x} \). Since

\[ \alpha + \beta + \theta = 180^\circ = \pi, \theta = \pi - \tan^{-1} \left( \frac{5}{x} \right) - \tan^{-1} \left( \frac{2}{3-x} \right) \]

\[
\frac{d\theta}{dx} = -\frac{1}{1 + \left( \frac{5}{x} \right)^2} \left( -\frac{5}{x^2} \right) - \frac{1}{1 + \left( \frac{2}{3-x} \right)^2} \left[ -\frac{2}{(3-x)^2} \right]
\]

\[
= \frac{x^2}{x^2 + 25} \cdot \frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2 + 4} \cdot \frac{2}{(3-x)^2}
\]

Now \( \frac{d\theta}{dx} = 0 \) \( \Rightarrow \frac{5}{x^2 + 25} = \frac{2}{x^2 - 6x + 13} \) \( \Rightarrow 2x^2 + 50 = 5x^2 - 30x + 65 \) \( \Rightarrow 3x^2 - 30x + 15 = 0 \) \( \Rightarrow x^2 - 10x + 5 = 0 \) \( \Rightarrow x = 5 \pm 2\sqrt{5} \). We reject the root with the + sign, since it is larger than 3. \( \frac{d\theta}{dx} > 0 \) for \( x < 5 - 2\sqrt{5} \) and \( \frac{d\theta}{dx} < 0 \) for \( x > 5 - 2\sqrt{5} \), so \( \theta \) is maximized when \( |AP| = x = 5 - 2\sqrt{5} \approx 0.53 \).