A function of two variables is a rule that assigns a real number \( f(x, y) \) to each ordered pair of real numbers \((x, y)\) in the domain of the function. For a function \( f \) defined on the domain \( D \subseteq \mathbb{R}^2 \), we sometimes write \( f : D \to \mathbb{R} \) to indicate that \( f \) maps points in two dimensions to real numbers. You may think of such a function as a rule whose input is a pair of real numbers and whose output is a single real number.

For instance, \( f(x, y) = xy^2 \) and \( g(x, y) = x^2 - e^y \) are both functions of the two variables \( x \) and \( y \).

A function of three variables is a rule that assigns a real number \( f(x, y, z) \) to each ordered triple of real numbers \((x, y, z)\) in the domain \( D \subseteq \mathbb{R}^3 \) of the function. We sometimes write \( f : D \to \mathbb{R} \) to indicate that \( f \) maps points in three dimensions to real numbers.

For instance, \( f(x, y, z) = xy^2 \cos z \) and \( g(x, y, z) = 3zx^2 - e^y \) are both functions of the three variables \( x, y \) and \( z \).

Unless specifically stated otherwise, the domain of a function of several variables is taken to be the set of all values of the variables for which the function is defined.
Find the domains of the following functions and evaluate \( f(3, 2) \).

(a) \( f(x, y) = \frac{\sqrt{x + y + 1}}{x - 1} \)  
(b) \( f(x, y) = x \ln(y^2 - x) \)

[Sol]:

(a) \[\begin{cases} 
  x + y + 1 \geq 0 \\
  x - 1 \neq 0 
\end{cases} \]  
\[\Rightarrow D_f = \{(x, y) | x + y + 1 \geq 0, x \neq 1\} \]

(b) \( y^2 - x > 0 \)  
\[\Rightarrow D_f = \{(x, y) | y^2 > x\} \]
Find the domain and range of \( g(x, y) = \sqrt{9 - x^2 - y^2} \).

**[Sol]:**

\[
D_f = \left\{ (x, y) \middle| 9 - x^2 - y^2 \geq 0 \right\} = \left\{ (x, y) \middle| x^2 + y^2 \leq 9 \right\}
\]

\[
R_f = \left\{ z \middle| z = \sqrt{9 - x^2 - y^2}, (x, y) \in D \right\} = \left\{ z \middle| 0 \leq z \leq 3 \right\} = [0, 3]
\]
[Ex] Find and sketch the domain for (a) \( f(x, y) = x \ln y \) and (b) \( g(x, y) = \frac{2x}{y - x^2} \).

[Sol]:
(a) For \( f(x, y) = x \ln y \), recall that \( \ln y \) is defined only for \( y > 0 \). The domain of \( f \) is then the set \( D = \{(x, y) | y > 0\} \), that is, the half-plane lying above the \( x \)-axis. (See Figure 1)

(b) For \( g(x, y) = \frac{2x}{y - x^2} \), note that \( g \) is defined unless there is a division by zero, which occurs when \( y - x^2 = 0 \). The domain of \( g \) is then \( \{(x, y) | y \neq x^2\} \), which is the entire \( xy \)-plane with the parabola \( y = x^2 \) removed. (see Figure 2)
[Ex] Find and describe in graphical terms (用圖形描述) the domains of

(a) \( f(x, y, z) = \frac{\cos(x + z)}{xy} \) and (b) \( g(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2} \).

[Sol]:

(a) For \( f(x, y, z) = \frac{\cos(x + z)}{xy} \), there is a division by zero if \( xy = 0 \), which occurs if \( x = 0 \) or \( y = 0 \). The domain is then \( \{(x, y, z) \mid x \neq 0 \text{ and } y \neq 0\} \), which is all of three-dimensional space excluding the \( yz \)-plane \( (x = 0) \) and the \( xz \)-plane \( (y = 0) \).

(b) Notice that for \( g(x, y, z) = \sqrt{9 - x^2 - y^2 - z^2} \) to be defined, you'll need to have \( 9 - x^2 - y^2 - z^2 \geq 0 \), or \( x^2 + y^2 + z^2 \leq 9 \). The domain of \( g \) is then the sphere (球體) of radius 3 centered at the origin and its interior.
The graph of a function of two variables is the graph of the equation \( z = f(x, y) \).

[Ex3,4][Ex5,6] Sketch the graph of the function (a) \( f(x, y) = 6 - 3x - 2y \) and (b) \( g(x, y) = \sqrt{9 - x^2 - y^2} \).

[Sol]:
(a) \( z = 6 - 3x - 2y \)
\[ \Rightarrow 3x + 2y + z = 6 \]

(b) \( z = \sqrt{9 - x^2 - y^2} \)
\[ \Rightarrow z^2 = 9 - x^2 - y^2 \]
\[ \Rightarrow x^2 + y^2 + z^2 = 9 \]
\[ \therefore z = \sqrt{9 - x^2 - y^2} \geq 0 \]
\[ \Rightarrow \text{The graph of the function is the top half sphere with center the origin and radius 3.} \]
[Ex] Graph (a) $f(x, y) = x^2 + y^2$ and (b) $g(x, y) = \sqrt{4 - x^2 + y^2}$.

[Sol]:
(a) For $f(x, y) = x^2 + y^2$, you may recognize the surface $z = x^2 + y^2$ as a circular paraboloid. Notice that the traces in the planes $z = k > 0$ are circles, while the traces in the planes $x = k$ and $y = k$ are parabolas. A graph is shown in Figure 1.
(b) For \( g(x, y) = \sqrt{4 - x^2 + y^2} \), note that the surface is the top half of the surface
\[ z^2 = 4 - x^2 + y^2 \text{ or } x^2 - y^2 + z^2 = 4. \]
Here, observe that the traces in the planes \( x = k \) and \( z = k \) are hyperbolas (雙曲線), while the traces in the planes \( y = k \) are circles. This gives us a hyperboloid of one sheet, wrapped around the \( y \)-axis.

The graph of \( z = g(x, y) \) is the top half of the hyperboloid, as shown in Figure 2.
(a) \( f(x, y) = (x^2 + 3y^2)e^{-x^2 - y^2} \)

(b) \( f(x, y) = (x^2 + 3y^2)e^{-x^2 - y^2} \)
(c) \( f(x, y) = \sin x + \sin y \)

(d) \( f(x, y) = \frac{\sin x \sin y}{xy} \)
Level Curves 等高線

**Def**
The *level curves* of a function $f$ of two variables are the curves with equations $f(x, y) = k$, where $k$ is a constant (常數) (in the range of $f$).
[Ex6][Ex9]

A contour map (等高線圖) for a function $f$ is shown. Use it to estimate the values $f(1,3)$ and $f(4,5)$.

[Sol]:

$$f(1,3) \approx 73 \quad f(4,5) \approx 56$$

[Ex7][Ex10]

Sketch the level curves of the function $f(x, y) = 6 - 3x - 2y$ for $k = -6, 0, 6, 12$.

[Sol]: The level curves are

$$6 - 3x - 2y = k \quad \Rightarrow 3x + 2y + (k - 6) = 0 \quad k = -6, 0, 6, 12.$$

$$\begin{align*}
3x + 2y - 12 &= 0 \\
3x + 2y - 6 &= 0 \\
3x + 2y &= 0 \\
3x + 2y + 6 &= 0
\end{align*}$$
Sketch the level curves of the function $g(x, y) = \sqrt{9 - x^2 - y^2}$ for $k = 0, 1, 2, 3$.

**Sol:**

The level curves are

$$\sqrt{9 - x^2 - y^2} = k, \ k = 0, 1, 2, 3$$

$$\Rightarrow x^2 + y^2 = 9 - k^2, \ k = 0, 1, 2, 3$$

$$\Rightarrow \begin{cases} 
  x^2 + y^2 = 9 \\
  x^2 + y^2 = 8 \\
  x^2 + y^2 = 5 \\
  x^2 + y^2 = 0 \quad i.e. \ (0, 0)
\end{cases}$$
[Ex9][Ex12] Sketch some level curves of the function \( h(x,y) = 4x^2 + y^2 \).

[Sol]:
The level curves are \( 4x^2 + y^2 = k \) or \( \frac{x^2}{k/4} + \frac{y^2}{k} = 1 \).
World mean sea-level temperatures in January in degrees Celsius
(a) Two views of \( f(x, y) = -xye^{-x^2-y^2} \)

(b) Level curves of \( f(x, y) = -xye^{-x^2-y^2} \)
(a) $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$

(b) Level curves of $f(x, y) = \frac{-3y}{x^2 + y^2 + 1}$
**Def**
The *level surfaces* (等高面) of a function \( f \) of three variables are the surfaces with equations \( f(x, y, z) = k \), where \( k \) is a constant (in the range of \( f \)).

[Ex11][Ex15] Find the level surfaces of the function \( f(x, y, z) = x^2 + y^2 + z^2 \).

[Sol]:
The level surfaces are \( x^2 + y^2 + z^2 = k \), where \( k \geq 0 \).

These are concentric spheres(同心球) centered the origin with radius \( \sqrt{k} \).
\[ \lim_{(x,y) \to (2,3)} (xy - 2) = 6 - 2 = 4 \]
\[ \lim_{(x,y) \to (-1,\pi)} (\sin xy - x^2 y) = \sin(-\pi) - \pi = -\pi \]

[Ex] Evaluate \[ \lim_{(x,y) \to (2,1)} \frac{2x^2 y + 3xy}{5xy^2 + 3y} \]

[Sol]:
\[ \therefore \lim_{(x,y) \to (2,1)} (5xy + 3y) = 10 + 3 = 13 \neq 0 \]
\[ \therefore \lim_{(x,y) \to (2,1)} \frac{2x^2 y + 3xy}{5xy^2 + 3y} = \frac{\lim_{(x,y) \to (2,1)} (2x^2 y + 3xy)}{\lim_{(x,y) \to (2,1)} (5xy^2 + 3y)} = \frac{14}{13}. \]
[Ex] Evaluate \( \lim_{(x,y) \to (1,0)} \frac{y}{x + y - 1} \)

[Sol]:

First, we consider the vertical line path along the line \( x = 1 \) and compute the limit as \( y \) approaches 0. If \((x, y) \to (1, 0) \) along the line \( x = 1 \), we have

\[
\lim_{(1,y) \to (1,0)} \frac{y}{x + y - 1} = \lim_{y \to 0} \frac{y}{1 + y - 1} = \lim_{y \to 0} 1 = 1
\]

We next consider the horizontal line \( y = 0 \) and compute the limit as \( x \) approaches 1. Here, we have

\[
\lim_{(x,0) \to (1,0)} \frac{y}{x + y - 1} = \lim_{x \to 0} \frac{0}{x + 0 - 1} = \lim_{x \to 0} 0 = 0
\]

Since the function is approaching two different values along two different paths to the point \((1, 0)\), the limit does not exist.
Evaluate \( \lim_{(x,y) \to (0,0)} \frac{xy}{x^2 + y^2} \)

**[Sol]:**

First, we consider the limit along the path \( x = 0 \). We have

\[
\lim_{(0,y) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{y \to 0} 0 = 0
\]

Similarly, for the path \( y = 0 \), we have

\[
\lim_{(x,0) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{y \to 0} 0 = 0
\]

Be careful; just because the limits along the first two paths you try are the same does **not** mean that the limit exists. Keep in mind that for a limit to exist, we'll need the limit to be the same along **all** paths through \((0, 0)\) (not just along two).

We simply need to look at more paths. Notice that for the path \( y = x \), we have

\[
\lim_{(x,x) \to (0,0)} \frac{xy}{x^2 + y^2} = \lim_{x \to 0} \frac{x(x)}{x^2 + x^2} = \lim_{x \to 0} \frac{x^2}{2x^2} = \frac{1}{2}
\]

Since the limit along this path doesn't match the limit along the first two paths, the limit does not exist.
**Def**

Suppose that $f(x, y)$ is defined in the interior of a circle centered at the point $(a, b)$. We say that $f$ is **continuous** at $(a, b)$ if

$$\lim_{(x, y) \to (a, b)} f(x, y) = f(a, b)$$

If $f(x, y)$ is not continuous at $(a, b)$, then we call $(a, b)$ a **discontinuity** of $f$.

**[Ex]** Where is the function $F(x, y) = \frac{4xy^2 + 8y}{x^2 - y}$ continuous?

**[Sol]**: 

$F$ is continuous on its domain: $\{(x, y) | y \neq x^2\}$
Recall that for a function $f$ of a single variable, we define the derivative function as $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, for any values of $x$ for which the limit exists.

If $f$ is a function of two variables and we hold one variable fixed ($y = b$), the following limit is call this the partial derivative of $f$ with respect to $x$:

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$
We hold another variable fixed \((x = a)\), the following limit is call this the \textit{partial derivative of} \(f\) \textit{with respect to} \(y\):

\[
\frac{\partial f}{\partial y}(a, b) = \lim_{h \to 0} \frac{f(a, b + h) - f(a, b)}{h}
\]
Def

(1) The **partial derivative of \( f(x, y) \) with respect to \( x \)**, written \( \frac{\partial f}{\partial x} \), is defined by

\[
\frac{\partial f}{\partial x}(x, y) = \lim_\limits{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}
\]

for any values of \( x \) and \( y \) for which the limit exists.

(2) The **partial derivative of \( f(x, y) \) with respect to \( y \)**, written \( \frac{\partial f}{\partial y} \), is defined by

\[
\frac{\partial f}{\partial y}(x, y) = \lim_\limits{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}
\]

for any values of \( x \) and \( y \) for which the limit exists.

Notations (符號)

For \( z = f(x, y) \), we write

\[
f_x(x, y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial z}{\partial x} \quad (\text{the value of } y \text{ is held constant})
\]

The expression \( \frac{\partial}{\partial x} \) is a **partial differential operator**.

\[
f_y(x, y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial y}(x, y) = \frac{\partial f}{\partial y}(x, y) = \frac{\partial z}{\partial y} \quad (\text{the value of } x \text{ is held constant})
\]
[Ex2][Ex2]

If \( f(x, y) = 4 - x^2 - 2y^2 \), find \( f_x(1, 1) \) and \( f_y(1, 1) \) and interpret these numbers as slopes.

[Sol]:

\[
\begin{align*}
  f_x(x, y) &= -2x \quad \Rightarrow \quad f_x(1, 1) = -2 \\
  f_y(x, y) &= -4y \quad \Rightarrow \quad f_y(1, 1) = -4
\end{align*}
\]
[Ex] If \( f(x, y) = e^{xy} + \frac{x}{y} \) , compute \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

[Sol]:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \left( e^{xy} + \frac{x}{y} \right) = e^{xy} y + \frac{1}{y} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \left( e^{xy} + \frac{x}{y} \right) = e^{xy} x - \frac{x}{y^2}
\end{align*}
\]

[Ex3][Ex3] If \( f(x, y) = \sin \left( \frac{x}{1 + y} \right) \) , compute \( \frac{\partial f}{\partial x} \) and \( \frac{\partial f}{\partial y} \).

[Sol]:
\[
\begin{align*}
\frac{\partial f}{\partial x} &= \frac{\partial}{\partial x} \left( \sin \left( \frac{x}{1 + y} \right) \right) = \cos \left( \frac{x}{1 + y} \right) \cdot \frac{x}{1 + y} = \cos \left( \frac{x}{1 + y} \right) \cdot \frac{1}{1 + y} \\
\frac{\partial f}{\partial y} &= \frac{\partial}{\partial y} \left( \sin \left( \frac{x}{1 + y} \right) \right) = \cos \left( \frac{x}{1 + y} \right) \cdot \frac{x}{1 + y} = \cos \left( \frac{x}{1 + y} \right) \cdot \frac{-x}{(1 + y)^2}
\end{align*}
\]
Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z$ is defined implicitly as a function of $x$ and $y$ by the equation $x^3 + y^3 + z^3 + 6xyz = 1$.

[Sol]:

(1) \[ \frac{\partial}{\partial x} (x^3 + y^3 + z^3 + 6xyz) = \frac{\partial}{\partial x} (1) \]

$\Rightarrow$ \[ 3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0 \]

$\Rightarrow \frac{\partial z}{\partial x} = -\frac{3x^2 + 6yz}{z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$

(2) \[ \frac{\partial}{\partial y} (x^3 + y^3 + z^3 + 6xyz) = \frac{\partial}{\partial y} (1) \]

$\Rightarrow$ \[ 0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 6xz + 6xy \frac{\partial z}{\partial y} = 0 \]

$\Rightarrow \frac{\partial z}{\partial y} = -\frac{3y^2 + 6xz}{z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$
[Ex5][Ex5] Find $f_x$, $f_y$ and $f_z$ if $f(x, y, z) = e^{xy} \ln z$

[Sol]:

\[
\begin{align*}
  f_x &= ye^{xy} \ln z \\
  f_y &= xe^{xy} \ln z \\
  f_z &= \frac{e^{xy}}{z}
\end{align*}
\]

**Exercise**

(1) Find $f_x$ and $f_y$ if $f(x, y) = \ln(x^2 + xy^3)$

(2) Find $f_x$ and $f_y$ if $f(x, y) = y^3 e^{2x+4y}$
If \( z = f(x, y) \), the \textit{second partial derivatives of} \( f \) (二階偏導數) are

\[
(f_x)_x = f_{xx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}
\]

\[
(f_x)_y = f_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x}
\]

\[
(f_y)_x = f_{yx} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 z}{\partial x \partial y}
\]

\[
(f_y)_y = f_{yy} = \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}
\]
Find the second partial derivatives of \( f(x, y) = x^3 + x^2 y^3 - 2y^2 \)

\[ f_x = 3x^2 + 2xy^3 \quad \Rightarrow \quad \begin{cases} f_{xx} = \frac{\partial}{\partial x} \left( 3x^2 + 2xy^3 \right) = 6x + 2y^3 \\ f_{xy} = \frac{\partial}{\partial y} \left( 3x^2 + 2xy^3 \right) = 6xy^2 \end{cases} \]

\[ f_y = 3x^2 y^2 - 4y \quad \Rightarrow \quad \begin{cases} f_{yx} = \frac{\partial}{\partial x} \left( 3x^2 y^2 - 4y \right) = 6xy^2 \\ f_{yy} = \frac{\partial}{\partial y} \left( 3x^2 y^2 - 4y \right) = 6x^2 y - 4 \end{cases} \]
[Ex7][Ex7] Calculate $f_{xyz}$, if $f(x, y, z) = \sin(3x + yz)$.

[Sol]:

$$f_x = 3 \cos(3x + yz)$$
$$f_{xx} = -9 \sin(3x + yz)$$
$$f_{xy} = -9z \cos(3x + yz)$$
$$f_{xyz} = -9 \cos(3x + yz) + 9yz \sin(3x + yz)$$

Exercise

Find $f_{xx}$ and $f_{xy}$ if $f(x, y) = y^3 e^{2x+4y}$
§ 11.4 (14.4) Tangent Planes and Linear Approximations

(切平面)

Tangent line equation:

\[ y = f(a) + f'(a)(x - a) \]

\[ z = 6 - x^2 - y^2 \text{ and the tangent plane at (1, 2, 1).} \]

tangent plane equation?
The curve lies in the plane $y = b$, whose slope at $x = a$ is given by $f_x(a, b)$.
A vector with the same direction as the tangent line at $x = a$ is $\langle 1, 0, f_x(a, b) \rangle$.

The curve lies in the plane $x = a$, whose slope at $y = b$ is given by $f_y(a, b)$.
A vector with the same direction as the tangent line at $y = b$ is $\langle 0, 1, f_y(a, b) \rangle$. 
We have now found two vectors in the tangent plane:
\[ \langle 1, 0, f_x(a,b) \rangle \text{ and } \langle 0, 1, f_y(a,b) \rangle . \]
A vector normal to the plane is then given by the cross product:
\[ \langle 0, 1, f_y(a,b) \rangle \times \langle 1, 0, f_x(a,b) \rangle = \langle f_x(a,b), f_y(a,b), -1 \rangle \]
Therefore, the equation of the tangent plane is
\[ \langle f_x(a,b), f_y(a,b), -1 \rangle \cdot \langle x - a, y - b, z - f(a,b) \rangle = 0 \]
i.e. \[ z = f(a,b) + f_x(a,b)(x - a) + f_y(a,b)(y - b) \]
**Theorem**
Suppose that \( f(x, y) \) has continuous partial derivatives at \((a, b)\). A normal vector to the tangent plane to \( z = f(x, y) \) at \((a, b)\) is \( \langle f_x(a, b), f_y(a, b), -1 \rangle \).
An equation of the tangent plane is given by
\[
z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)
\]

[Ex] Find equations of the tangent plane and the normal line to \( z = 6-x^2-y^2 \) at the point \((1, 2, 1)\).

[Sol]:
Let \( f(x, y) = 6-x^2-y^2 \), then \( f_x = -2x \) and \( f_y = -2y \).
So \( f_x(1, 2) = -2 \) and \( f_y(1, 2) = -4 \).
A normal vector is \( \langle -2, -4, -1 \rangle \).
The equation of the tangent plane is
\[
z = 1 - 2(x - 1) - 4(y - 2)
\]
The equations of the normal line are
\[
x = 1 - 2t, \quad y = 2 - 4t, \quad z = 1 - t.
\]
§ 11.5 (14.5) The Chain Rule (連鎖法則)

If \( y = f(x) \) and \( x = g(t) \), where \( f \) and \( g \) are differentiable. Then

\[
\frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt}
\]
(The Chain Rule for single variable functions)

[Ex1][Ex1] If \( z = x^2y + 3xy^4 \), where \( x = \sin2t \), \( y = \cos t \), find \( \frac{dz}{dt} \) when \( t = 0 \).

[Sol]:

\[
\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}
\]

\[
= (2xy + 3y^4)(2 \cos 2t) + (x^2 + 12xy^3)(-\sin t)
\]

When \( t = 0 \), \( x = \sin 0 = 0 \) and \( y = \cos 0 = 1 \).

Therefore,

\[
\left. \frac{dz}{dt} \right|_{t=0} = (0 + 3)(2 \cos 0) + (0 + 0)(-\sin 0) = 6
\]
If $z = e^x \sin y$, where $x = st^2$, $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

[Sol]:

\[
\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial s} = (e^x \sin y)(t^2) + (e^x \cos y)(2st) \\
= t^2 e^{st^2} \sin(s^2 t) + 2ste^{st^2} \cos(s^2 t)
\]

\[
\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial t} = (e^x \sin y)(2st) + (e^x \cos y)(s^2) \\
= 2ste^{st^2} \sin(s^2 t) + s^2 e^{st^2} \cos(s^2 t)
\]

[Ex4]

Write out the Chain Rule for $\frac{\partial w}{\partial u}$ in the case when $w = f(x, y, z, t)$ and $x = x(u, v), y = y(u, v), z = z(u, v)$, and $t = t(u, v)$. 
If \( u = x^4y + y^2z^3 \), where \( x = rse^t \), \( y = rs^2e^{-t} \), and \( z = r^2s \sin t \), find the value of \( \frac{\partial u}{\partial s} \) when \( r = 2 \), \( s = 1 \), \( t = 0 \).

[Sol]:

\[
\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial s} \\
= (4x^3y)(re^t) + (x^4 + 2yz^3)(2rs^{-t}) + (3y^2z^2)(r^2 \sin t)
\]

When \( r = 2 \), \( s = 1 \), \( t = 0 \), we have \( x = 2 \), \( y = 2 \) and \( z = 0 \).

So

\[
\frac{\partial u}{\partial s}\bigg|_{\substack{r=2 \\ s=1 \\ t=0}} = (64)(2) + (16)(4) + (0)(0) = 192
\]

**Exercise**

Write out the Chain Rule for the case where \( w = f(x, y) \), \( x = x(u, v) \), \( y = y(u, v) \), and \( u = u(s, t) \), and \( v = v(s, t) \).
If \( g(s, t) = f(s^2 - t^2, t^2 - s^2) \), and \( f \) is differentiable, show that \( g \) satisfies the equation \( t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = 0 \).

[Sol]:

Let \( x = s^2 - t^2 \) and \( y = t^2 - s^2 \).

Then \( g(s, t) = f(x, y) \) and the Chain Rule gives

\[
\frac{\partial g}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} = \frac{\partial f}{\partial x} (2s) + \frac{\partial f}{\partial y} (-2s) \\
\frac{\partial g}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} = \frac{\partial f}{\partial x} (2t) + \frac{\partial f}{\partial y} (-2t)
\]

Therefore

\[
t \frac{\partial g}{\partial s} + s \frac{\partial g}{\partial t} = \left( 2st \frac{\partial f}{\partial x} - 2st \frac{\partial f}{\partial y} \right) + \left( -2st \frac{\partial f}{\partial x} + 2st \frac{\partial f}{\partial y} \right) = 0
\]
If \( z = f(x, y) \) has continuous second-order partial derivatives and \( x = r^2 + s^2 \) and \( y = 2rs \), find (a) \( \frac{\partial z}{\partial r} \) and (b) \( \frac{\partial^2 z}{\partial r^2} \)

[Sol]:

(a) The Chain Rule gives
\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} = \frac{\partial z}{\partial x} (2r) + \frac{\partial z}{\partial y} (2s)
\]

(b) Applying the Product Rule to the expression in part (a), we get
\[
\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} \left( 2r \frac{\partial z}{\partial x} + 2s \frac{\partial z}{\partial y} \right) = 2 \frac{\partial z}{\partial x} + 2r \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) + 2s \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right)
\] (*)
But, using the Chain Rule again, we have

\[
\frac{\partial}{\partial r} \left( \frac{\partial z}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial x} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right) \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x^2} (2r) + \frac{\partial^2 z}{\partial y \partial x} (2s)
\]

\[
\frac{\partial}{\partial r} \left( \frac{\partial z}{\partial y} \right) = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial y} \right) \frac{\partial y}{\partial r} = \frac{\partial^2 z}{\partial x \partial y} (2r) + \frac{\partial^2 z}{\partial y^2} (2s)
\]

Putting these expression into Equation (*), we obtain

\[
\frac{\partial^2 z}{\partial r^2} = 2 \frac{\partial z}{\partial x} + 2r \left(2r \frac{\partial^2 z}{\partial x^2} + 2s \frac{\partial^2 z}{\partial y \partial x} \right) + 2s \left(2r \frac{\partial^2 z}{\partial x \partial y} + 2s \frac{\partial^2 z}{\partial y^2} \right)
\]

\[
= 2 \frac{\partial z}{\partial x} + 4r^2 \frac{\partial^2 z}{\partial x^2} + 8rs \frac{\partial^2 z}{\partial x \partial y} + 4s^2 \frac{\partial^2 z}{\partial y^2}
\]
Implicit Differentiation (隱微分)

Suppose that the equation $F(x, y) = 0$ defines $y$ implicitly as a differentiable function of $x$. If $F$ is differentiable, we have

$$F(x, y) = 0 \implies \frac{d}{dx} F(x, y) = 0 \implies \frac{\partial F}{\partial x} \frac{dx}{dx} + \frac{\partial F}{\partial y} \frac{dy}{dx} = 0 \implies \frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

[Ex8][Ex8] Find $y'$ if $x^3 + y^3 = 6xy$.

[Sol]:

$$x^3 + y^3 = 6xy \implies x^3 + y^3 - 6xy = 0$$

Let $F(x, y) = x^3 + y^3 - 6xy$, then

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$
Suppose that the equation $F(x, y, z) = 0$ defines $z$ implicitly as a differentiable function of $x$ and $y$. If $F$ is differentiable,

$$F(x, y) = 0 \Rightarrow \frac{\partial}{\partial x} F(x, y, z) = 0 \Rightarrow \frac{\partial F}{\partial x} \frac{\partial x}{\partial x} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial x} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial x} = 0$$

Since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, we have

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

Similarly, if we take partial derivatives w.r.t. $y$, we get

$$F(x, y) = 0 \Rightarrow \frac{\partial}{\partial y} F(x, y, z) = 0 \Rightarrow \frac{\partial F}{\partial x} \frac{\partial x}{\partial y} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial y} + \frac{\partial F}{\partial z} \frac{\partial z}{\partial y} = 0$$

Since $\frac{\partial y}{\partial y} = 1$ and $\frac{\partial x}{\partial y} = 0$, we have

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$
[Ex9][Ex9] Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^3 + y^3 + z^3 + 6xyz = 1$.

[Sol]:

Let $F(x, y, z) = x^3 + y^3 + z^3 + 6xyz - 1$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} = -\frac{x^2 + 2yz}{z^2 + 2xy}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\frac{y^2 + 2xz}{z^2 + 2xy}$$
§ 11.6 (14.6) Directional Derivatives and the Gradient Vector

**Def**
The **directional derivative of** \( f \) **at** \( (x_0, y_0) \) **in the direction of a unit vector** \( u = \langle a, b \rangle \) **is**

\[
D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}
\]

if the limit exists.

Note that if \( u = i = \langle 1, 0 \rangle \), then \( D_i f = f_x \)

and if \( u = j = \langle 0, 1 \rangle \), then \( D_j f = f_y \).

In other words, the partial derivative of \( f \) w.r.t. \( x \) and \( y \) are just special cases of the directional derivatives.
**Theorem**

If \( f(x, y) \) is a differentiable function of \( x \) and \( y \), then \( f \) has a directional derivative in the direction of any unit vector \( u = \langle a, b \rangle \) and

\[
D_u f(x, y) = f_x(x, y)a + f_y(x, y)b
\]

[Ex]

For \( f(x, y) = x^2y - 4y^3 \), compute \( D_u f(2, 1) \) and \( u \) is in the direction from \((2, 1)\) to \((4, 0)\)

[Sol]:

The vector from \((2, 1)\) to \((4, 0)\) corresponds to the position vector \( \langle 4 - 2, 0 - 1 \rangle = \langle 2, -1 \rangle \) and \( |\langle 2, -1 \rangle| = \sqrt{5} \). So the unit vector in that direction is \( u = \frac{1}{\sqrt{5}} \langle 2, -1 \rangle = \left\langle \frac{2}{\sqrt{5}}, \frac{-1}{\sqrt{5}} \right\rangle \)

Since \( f_x(x, y) = 2xy \) and \( f_y(x, y) = x^2 - 12y^2 \), we have

\[
D_u f(2, 1) = f_x(2, 1) \frac{2}{\sqrt{5}} + f_y(2, 1) \frac{-1}{\sqrt{5}} = (4) \frac{2}{\sqrt{5}} + (-8) \frac{-1}{\sqrt{5}} = \frac{16}{\sqrt{5}}
\]
Find the directional derivative $D_u f(x, y)$ if $f(x, y) = x^3 - 3xy + 4y^2$ and $u$ is the unit vector given by angle $\theta = \pi / 6$. What is $D_u f(1, 2)$?

[Sol]:

$$D_u f(x, y) = f_x(x, y) \cos(\pi/6) + f_y(x, y) \sin(\pi/6)$$

$$= (3x^2 - 3y) \sqrt{3}/2 + (-3x + 8y)(1/2)$$

$$= \frac{1}{2} \left[ 3\sqrt{3}x^2 - 3x + (8 - 3\sqrt{3})y \right]$$

Therefore

$$D_u f(1, 2) = \frac{1}{2} \left[ 3\sqrt{3}(1)^2 - 3(1) + (8 - 3\sqrt{3})2 \right]$$

$$= \frac{13 - 3\sqrt{3}}{2}$$
The Gradient Vector

**Def**

If \( f \) is a function of two variables \( x \) and \( y \), then the *gradient of \( f \)* is the vector function \( \nabla f \) defined by

\[
\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = f_x(x, y)i + f_y(x, y)j
\]

[Ex2][Ex3]

If \( f(x, y) = \sin x + e^{xy} \), then

\[
\nabla f(x, y) = \left\langle f_x(x, y), f_y(x, y) \right\rangle = \left\langle \cos x + ye^{xy}, xe^{xy} \right\rangle
\]

and \( \nabla f(0, 1) = \left\langle 2, 0 \right\rangle \).

With the notation for the gradient vector, we have

\[
D_u f(x, y) = \nabla f(x, y) \cdot u
\]
Find the directional derivative of the function \( f(x, y) = x^2y^3 - 4y \) at the point (2, -1) in the direction of the vector \( v = 2i + 5j \).

**[Sol]:**
The gradient vector at (2, -1) is
\[
∇f(2, -1) = -4i + 8j
\]
Since \(|v| = \sqrt{29}\), the unit vector in the direction of the vector \( v \) is
\[
u = \frac{v}{|v|} = \frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j
\]
Therefore
\[
D_uf(2, -1) = ∇f(2, -1) \cdot u = (-4i + 8j) \cdot \left( \frac{2}{\sqrt{29}} i + \frac{5}{\sqrt{29}} j \right) = \frac{-4 \cdot 2 + 8 \cdot 5}{\sqrt{29}} = \frac{32}{\sqrt{29}}
\]

**Exercise**
Find the directional derivative of the function \( f(x, y) = \sqrt{xy} \) at the point \( P(2, 8) \) in the direction of \( Q(5, 4) \).
**Functions of Three Variables** 三個變數的函數

**Def**

The *directional derivative of* \( f \) *at* \( (x_0, y_0, z_0) \) in the direction of a unit vector \( u = \langle a, b, c \rangle \) is

\[
D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}
\]

if the limit exists.

For a function \( f \) of three variables, the **gradient vector**, denoted by \( \nabla f \) or **grad** \( f \), is

\[
\nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle
\]

or,

\[
\nabla f = \langle f_x, f_y, f_z \rangle = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}
\]

So

\[
D_u f(x, y, z) = \nabla f(x, y, z) \cdot u
\]
If \( f(x, y, z) = x \sin yz \), (a) find the gradient of \( f \) and (b) find the directional derivative of \( f \) at \((1, 3, 0)\) in the direction of the vector \( v = i + 2j - k \).

[Sol]:

(a) \[ \nabla f(x, y, z) = \left( f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \right) = \left( \sin yz, xz \cos yz, xy \cos yz \right) \]

(b) The unit vector in the direction of the vector \( v \) is

\[ u = \frac{v}{\|v\|} = \frac{1}{\sqrt{6}} (i+2j-k) = \left( \frac{1}{\sqrt{6}} i + \frac{2}{\sqrt{6}} j - \frac{1}{\sqrt{6}} k \right) \]

Therefore

\[ D_u f(1, 3, 0) = \nabla f(1, 3, 0) \cdot u = (3k) \cdot \left( \frac{1}{\sqrt{6}} i + \frac{2}{\sqrt{6}} j - \frac{1}{\sqrt{6}} k \right) = 3 \left( -\frac{1}{\sqrt{6}} \right) = -\frac{3}{\sqrt{2}} \]
**Maximizing the Directional Derivative** (方向導數的極大值)

**Theorem**

Suppose that $f$ is differentiable at the point $(x_0, y_0)$. Then

(i) the maximum rate of change of $f$ at $(x_0, y_0)$ is $|\nabla f(x_0, y_0)|$ and occurs in the direction of the gradient $\nabla f(x_0, y_0)$, $u = \frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$;

(ii) the minimum rate of change of $f$ at $(x_0, y_0)$ is $-|\nabla f(x_0, y_0)|$ and occurs in the direction opposite the gradient $-\nabla f(x_0, y_0)$, $u = -\frac{\nabla f(x_0, y_0)}{|\nabla f(x_0, y_0)|}$; and

(iii) the gradient $\nabla f(x_0, y_0)$ is orthogonal (正交) to the level curve $f(x, y) = k$ at the point $(x_0, y_0)$, where $k = f(x_0, y_0)$. 
\[ D_u f(x, y) = \nabla f(x, y) \cdot u = \left| \nabla f \right| |u| \cos \theta = \left| \nabla f \right| \cos \theta \]

So the maximum value of \( D_u f \) is \( \left| \nabla f \right| \), and it occurs when \( \theta = 0 \), that is, when \( u \) has the same direction as \( \nabla f \).

When \( \theta = \pi/2 \) (i.e. \( u \) is perpendicular to \( \nabla f \)), \( D_u f = 0 \) which means \( u \) is tangent to a level curve because the level curves are curves in the \( xy \)-plane on which \( f \) is constant. We therefore conclude that the vector \( \nabla f \) is orthogonal to the level curve \( f(x, y) = k \).
[Ex]
Find the maximum and minimum rates of change of the function $f(x, y) = x^2 + y^2$
at the point $(1, 3)$.

[Sol]:
\[
\nabla f(x, y) = \langle 2x, 2y \rangle \quad \Rightarrow \quad \nabla f(1, 3) = \langle 2, 6 \rangle
\]
So the maximum rate of change of $f$ at $(1, 3)$ is $|\nabla f(1, 3)| = |\langle 2, 6 \rangle| = \sqrt{40}$,
and occurs in the direction $\nabla f(1, 3) = \langle 2, 6 \rangle$, $u = \frac{\langle 2, 6 \rangle}{|\langle 2, 6 \rangle|} = \left\langle \frac{2}{\sqrt{40}}, \frac{6}{\sqrt{40}} \right\rangle$.

Similarly, the minimum rate of change is $-|\nabla f(1, 3)| = -|\langle 2, 6 \rangle| = -\sqrt{40}$,
which occurs in the direction $-\nabla f(1, 3) = -\langle 2, 6 \rangle = \langle -2, -6 \rangle$,
\[
\quad u = \frac{-\langle 2, 6 \rangle}{|\langle 2, 6 \rangle|} = \left\langle \frac{-2}{\sqrt{40}}, \frac{-6}{\sqrt{40}} \right\rangle
\]
(a) If $f(x, y) = xe^y$, find the rates of change of $f$ at the point $P(2, 0)$ in the direction from $P$ to $Q (1/2, 2)$.

(b) In what direction does $f$ have the maximum rate of change? What is this maximum rate of change?

[Sol]:
(a) \[ \nabla f(x, y) = \langle e^y, xe^y \rangle \Rightarrow \nabla f(2, 0) = \langle 1, 2 \rangle \]
The unit vector in the direction of $\overrightarrow{PQ} = \langle -1.5, 2 \rangle$ is
\[ u = \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle, \]
So the rate of change of $f$ in the direction from $P$ to $Q$ is
\[ D_u f(2, 0) = \nabla f(2, 0) \cdot u = \langle 1, 2 \rangle \cdot \left\langle \frac{-3}{5}, \frac{4}{5} \right\rangle = 1 \]

(b) $f$ have the maximum rate of change in what direction of the gradient vector, $\nabla f(2, 0) = \langle 1, 2 \rangle$ .
The maximum rate of change is
\[ |\nabla f(2, 0)| = |\langle 1, 2 \rangle| = \sqrt{5} \]
[Ex]
The contour plot (等高線圖) of \( f(x, y) = 3x-x^3-3xy^2 \) shown in the graph indicates several level curves near a relative maximum at \( (1, 0) \). Find the direction of maximum increase from the point \( A(0.6, -0.7) \) and sketch in the path of steepest ascent (上升).

[Sol]:
The direction of maximum increase at \( (0.6, -0.7) \) is given by the gradient \( \nabla f(0.6, -0.7) \).

\[
\nabla f(x, y) = \left\langle 3 - 3x^2 - 3y^2, -6xy \right\rangle
\]

So, \( \nabla f(0.6, -0.7) = \langle 0.45, 2.52 \rangle \). The path of steepest ascent is a curve that remains perpendicular (垂直) to each level curve it passes through.
Find the equations of the tangent plane and normal line at the point \((-2, 1, -3)\) to ellipsoid (橢圓體) \(\frac{x^2}{4} + y^2 + \frac{z^2}{9} = 3\).

[Sol]:
The ellipsoid is a level surface (with \(k=3\)) of the function \(F(x, y, z) = \frac{x^2}{4} + y^2 + \frac{z^2}{9}\), a normal vector to the tangent plane at the point \((-2, 1, -3)\) is given by \(\nabla F(-2,1,-3)\).

We have \(\nabla F(x, y, z) = \langle \frac{x}{2}, 2y, \frac{2z}{9} \rangle\) and \(\nabla F(-2,1,-3) = \langle -1, 2, -2/3 \rangle\).

Given the normal vector \(\langle -1, 2, -2/3 \rangle\) and point \((-2, 1, -3)\), an equation of the tangent plane is

\[-1(x + 2) + 2(y - 1) - \frac{2}{3}(z + 3) = 0\]

or

\[3x - 6y + 2z + 18 = 0\]

The equations of the normal line are

\[\frac{x + 2}{-1} = \frac{y - 1}{2} = \frac{z + 3}{-2/3}\]
[Ex]
Find an equation of the tangent plane to \( x^3y - y^2 + z^2 = 7 \) at the point \((1, 2, 3)\).

[Sol]:

If we interpret the surface as a level surface of the function \( F(x, y, z) = x^3y - y^2 + z^2 \), a normal vector to the tangent plane at the point \((1, 2, 3)\) is given by \( \nabla F(1, 2, 3) \). We have \( \nabla F(x, y, z) = \langle 3x^2y, x^3 - 2y, 2z \rangle \) and \( \nabla F(1, 2, 3) = \langle 6, -3, 6 \rangle \). Given the normal vector \( \langle 6, -3, 6 \rangle \) and a point \((1, 2, 3)\), an equation of the tangent plane is \( 6(x - 1) - 3(y - 2) + 6(z - 3) = 0 \).
§ 11.7 (14.7) Maximum and Minimum Values

Def
The point \((a, b)\) is a **critical point** of the function \(f(x, y)\) if \((a, b)\) is in the domain of \(f\) and either \(f_x(a, b) = f_y(a, b) = 0\) or one or both of \(f_x\) and \(f_y\) do not exist at \((a, b)\).

Theorem
If \(f(x, y)\) has a local extremum at \((a, b)\), then \((a, b)\) must be a critical point of \(f\).
**Theorem (Second Derivatives Test)**

Suppose that $f(x, y)$ has continuous second-order partial derivatives in some open disk containing the point $(a, b)$ and that $f_x(a, b) = f_y(a, b) = 0$. Define the **discriminant** $D$ for the point $(a, b)$ by:

$$D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2.$$

(i) If $D(a, b) > 0$ and $f_{xx}(a, b) > 0$, then $f$ has a local minimum at $(a, b)$.

(ii) If $D(a, b) > 0$ and $f_{xx}(a, b) < 0$, then $f$ has a local maximum at $(a, b)$.

(iii) If $D(a, b) < 0$, then $f$ has a saddle point (鞍点) at $(a, b)$.

(iv) If $D(a, b) = 0$, then no conclusion can be drawn.

**Note 1:**

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

**Note 2:** In case (iv), if $D(a, b) = 0$, the test gives no information: $f$ could have a local maximum or a local minimum at $(a, b)$, or $(a, b)$ could be a saddle point of $f$. 
Note 3: In case (iii) the point \((a, b)\) is called a **saddle point** of \(f\) and the graph of \(f\) crosses its tangent plane at \((a, b)\).
Find the local maximum and minimum values and saddle points of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$

[Solution]:

1. Find critical points:
   \[
   \begin{cases}
   f_x = 2x - 2 = 0 \\
   f_y = 2y - 6 = 0
   \end{cases}
   \Rightarrow x = 1, y = 3
   
   So the only critical point is (1, 3).

2. \[f_{xx} = 2, \quad f_{xy} = 0 \quad \text{and} \quad f_{yy} = 2,\]
   \[D(x, y) = (2)(2) - (0)^2 = 4\]
   \[D(1,3) = 4 > 0 \quad \text{and} \quad f_{xx}(1,1) = 2 > 0\]
   \[\Rightarrow \quad f(1,3) = 4 \text{ is a local minimum value.}\]

   $f$ has no local maximum values

   \[
   f(x, y) = x^2 + y^2 - 2x - 6y + 14 = (x-1)^2 + (y-3)^2 + 4
   \]

   \[\Rightarrow \quad f(1,3) = 4 \text{ is the absolute (and local) minimum value.}\]
Find the local maximum and minimum values and saddle points of
\[ f(x, y) = x^4 + y^4 - 4xy + 1 \]

[Sol ]:

(1) Find critical points:
\[
\begin{align*}
  f_x &= 4x^3 - 4y = 0 \\
  f_y &= 4y^3 - 4x = 0
\end{align*}
\]
\[ \Rightarrow x = -1, 0, 1 \]
So the three critical points are (–1, –1), (0, 0), and (1, 1).

(2) \( f_{xx} = 12x^2, \ f_{xy} = -4 \) and \( f_{yy} = 12y^2 \),
\[ \therefore D(x, y) = (12x^2)(12y^2) - (-4)^2 = 144x^2y^2 - 16 \]

(i) \( D(1,1) = 128 > 0 \) and \( f_{xx}(1,1) = 12 > 0 \)
\[ \Rightarrow f(1, 1) = -1 \ is \ a \ local \ minimum. \]

(ii) \( D(0,0) = -16 < 0 \Rightarrow (0,0) \) is a saddle point;
\[ i.e. \ f \ has \ no \ local \ maximum \ or \ minimum \ at \ (0,0). \]

(iii) \( D(-1,-1) = 128 > 0 \) and \( f_{xx}(-1,-1) = 12 > 0 \)
\[ \Rightarrow f(-1, -1) = -1 \ is \ a \ local \ minimum. \]
Find and classify the critical points of the function

\[ f(x, y) = 10x^2 y - 5x^2 - 4y^2 - x^4 - 2y^4 \]

**[Sol ]:**

(1) \( f_x = 20xy - 10x - 4x^3 \), \( f_y = 10x^2 - 8y - 8y^3 \)

Solving \( \begin{cases} 20xy - 10x - 4x^3 = 0 \\ 10x^2 - 8y - 8y^3 = 0 \end{cases} \) for \( x \) and \( y \), we have \( x \approx \pm 0.8567, \ y \approx \pm 2.6442 \) and \( x = 0 \).

So the points \((\pm 0.8567, 0.6468), (\pm 2.6442, 1.8984)\) and \((0,0)\) are critical points.

(2) \( f_{xx} = 20y - 10 - 12x^2 \), \( f_{xy} = 20x \) and \( f_{yy} = -8 - 24y^2 \),

\[ D(x, y) = (20y - 10 - 12x^2)(-8 - 24y^2) - (20x)^2 \]

<table>
<thead>
<tr>
<th>Critical points</th>
<th>( D )</th>
<th>( f_{xx} )</th>
<th>Value of ( f )</th>
<th>Conclusion</th>
</tr>
</thead>
<tbody>
<tr>
<td>((0,0))</td>
<td>80</td>
<td>-10</td>
<td>0</td>
<td>local maximum</td>
</tr>
<tr>
<td>((\pm 2.64, 1.90))</td>
<td>2488.71</td>
<td>-55.93</td>
<td>8.5</td>
<td>local maximum</td>
</tr>
<tr>
<td>((\pm 0.86, 0.65))</td>
<td>-187.64</td>
<td>-5.87</td>
<td>-1.48</td>
<td>saddle points</td>
</tr>
</tbody>
</table>
Find the shortest distance from the point \((1, 0, -2)\) to the plane \(x + 2y + z = 4\).

**[Sol ]:**

The distance from any point \((x, y, z)\) on the plane \(x + 2y + z = 4\) to the point \((1, 0, -2)\) is

\[
d = \sqrt{(x-1)^2 + y^2 + (z+2)^2}
\]

and \(x + 2y + z = 4\)

\[
\therefore z = 4 - x - 2y \quad \therefore d = \sqrt{(x-1)^2 + y^2 + (6-x-2y)^2}
\]

We can minimize \(d\) by minimizing \(d^2\) which is much simpler.

Let \(f(x, y) = d^2 = (x-1)^2 + y^2 + (6-x-2y)^2\)

Solving \[\begin{cases} f_x(x, y) = 2(x-1) - 2(6-x-2y) = 0 \\ f_y(x, y) = 2y - 4(6-x-2y) = 0 \end{cases}\]

we find the only critical point \(\left(\frac{11}{6}, \frac{5}{3}\right)\).

Since \(f_{xx} = 4, f_{xy} = 4, \text{ and } f_{yy} = 10\), we have \(D(x, y) = f_{xx} f_{yy} - (f_{xy})^2 = 24 > 0\).

So \(f\) has a local minimum at \((11/6, 5/3)\). Intuitively, it is actually an absolutely minimum.

Because there must be a point on the given plane that is closest to the point \((1, 0, -2)\).

When \(x = \frac{11}{6}\) and \(y = \frac{5}{3}\),

\[
d = \sqrt{\left(\frac{11}{6} - 1\right)^2 + \left(\frac{5}{3}\right)^2 + \left(6 - \frac{11}{6} - 2\left(\frac{5}{3}\right)\right)^2} = \sqrt{\left(\frac{5}{6}\right)^2 + \left(\frac{5}{3}\right)^2 + \left(\frac{5}{3}\right)^2} = \frac{5\sqrt{6}}{6}
\]
Absolute Maximum and Minimum Values

To find absolute maximum and minimum of a continuous function \( f \) on a closed and bounded region (封閉有界區域) \( D \).

(i) Evaluate \( f \) at critical points in the interior of the region \( D \).

(ii) Find the maximum and minimum values of \( f \) on the boundary of \( D \).

(iii) Compare the values of \( f \) at the critical points with the maximum and minimum values of \( f \) on the boundary of \( D \).

[Ex6][Ex7]

Find the absolute maximum and minimum values of \( f(x, y) = x^2 - 2xy + 2y \) on the rectangle \( D = \{(x, y) | 0 \leq x \leq 3, 0 \leq y \leq 2\} \)

[Sol ]:

(1) In the interior of \( D \):

\[
\begin{align*}
    f_x &= 2x - 2y = 0 \\
    f_y &= -2x + 2 = 0
\end{align*}
\]

\( \Rightarrow (x, y) = (1, 1) \) is the only critical point in the interior of \( D \).

\[ f(1,1) = 1. \]
(2) On the boundary of $D$ (D的邊界上的極值)

(i) On $L_1$: $(y = 0$ and $0 \leq x \leq 3)$

\[ f(x, y) = f(x, 0) = x^2 \] which is increasing on $[0,3]$.

\[ \Rightarrow \begin{cases} f(0,0) = 0 & \text{is the absolute minimum value on } L_1. \\ f(3,0) = 9 & \text{is the absolute maximum value on } L_1. \end{cases} \]

(ii) On $L_2$: $(x = 3$ and $0 \leq y \leq 2)$

\[ f(x, y) = f(3, y) = 9 - 4y \] which is decreasing on $[0,2]$.

\[ \Rightarrow \begin{cases} f(3,2) = 1 & \text{is the absolute minimum value on } L_2. \\ f(3,0) = 9 & \text{is the absolute maximum value on } L_2. \end{cases} \]

(iii) On $L_3$: $(y = 2$ and $0 \leq x \leq 3)$

\[ f(x, y) = f(x, 2) = x^2 - 4x + 4 = (x - 2)^2 \Rightarrow f(2,2) = 0 \]

is the absolute minimum value on $L_3$

\[ \therefore f(0,2) = 4 > f(3,2) = 1 \quad \therefore f(0,2) = 4 \text{ is the absolute maximum value on } L_3. \]

(iv) On $L_4$: $(x = 0$ and $0 \leq y \leq 2)$

\[ f(x, y) = f(0, y) = 2y \] which is increasing on $[0,2]$.

\[ \Rightarrow \begin{cases} f(0,0) = 0 & \text{is the absolute minimum value on } L_4. \\ f(0,2) = 4 & \text{is the absolute maximum value on } L_4. \end{cases} \]
(3) From (2), we know that

\[
\begin{align*}
    f(0,0) &= 0 = \text{the absolute minimum value of } f \text{ on the boundary of } D \\
    f(3,0) &= 9 = \text{the absolute minimum value of } f \text{ on the boundary of } D
\end{align*}
\]

Comparing these values with \( f(1,1) = 1 \) from (1), we conclude that

\[
\begin{align*}
    f(0,0) &= 0 \text{ is the absolute minimum value on } D. \\
    f(3,0) &= 9 \text{ is the absolute maximum value on } D.
\end{align*}
\]
§ 11.8 (14.8) Lagrange Multipliers (拉格朗日乘數)

**Method of Lagrange Multipliers**  To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq 0$ on the surface $g(x, y, z) = k$]:

(a) Find all values of $x, y, z,$ and $\lambda$ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate $f$ at all the points $(x, y, z)$ that result from step (a). The largest of these values is the maximum value of $f$; the smallest is the minimum value of $f$.

Note: (1) The constant $\lambda$ in the method is called the Lagrange Multiplier.

(2) The equation $\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$ is equivalent to

$$f_x = \lambda g_x, \quad f_y = \lambda g_y, \quad \text{and} \quad f_z = \lambda g_z$$
[Ex1] A rectangular box without a lid (蓋子) is to be made from 12cm² of cardboard (硬紙板). Find the maximum volume of such a box.

[Sol]:
Let the length, width and height of the box be \(x, y, \) and \(z\) meters, respectively. Then we want to maximize the volume \(V(x, y, z) = xyz\) subject to the constraint \(g(x, y, z) = 2xz + 2yz + xy = 12\).

According to the Method of Lagrange Multiplier, we have to solve

\[
\begin{align*}
V_x &= \lambda g_x \\
V_y &= \lambda g_y \\
V_z &= \lambda g_z \\
2xz + 2yz + xy &= 12
\end{align*}
\]

\[
\begin{align*}
yz &= \lambda (2z + y) \\
xz &= \lambda (2z + x) \\
xy &= \lambda (2x + 2y) \\
2xz + 2yz + xy &= 12
\end{align*}
\]

Since \(V(\sqrt{12}, \sqrt{12}, 0) = 0 < V(2, 2, 1) = 4\),
we know that the maximum volume of such a box is 4 cm³.
Find the extreme values of the function \( f(x, y) = x^2 + 2y^2 \) on the circle \( x^2 + y^2 = 1 \).

[Sol]:

To find the max. and min. values of \( f(x, y) = x^2 + 2y^2 \) subject to the constraint \( g(x, y) = x^2 + y^2 = 1 \),
We use the Method of Lagrange Multiplier:

\[
\begin{align*}
\nabla f &= \lambda \nabla g \\
\Rightarrow \\
g(x, y) &= 1 \\
\Rightarrow \\
f_x &= \lambda g_x \\
f_y &= \lambda g_y \\
x^2 + y^2 &= 1
\end{align*}
\]

Therefore, \( f \) has possible extreme values at the points \((0,1), (0,-1), (1,0), \) and \((-1,0)\).

Since \( f(0,1) = 2, f(0,-1) = 2, f(1,0) = 1, f(-1,0) = 1 \),
the maximum value of \( f \) on the circle \( x^2 + y^2 = 1 \)
is \( f(0, \pm 1) = 2 \), and the minimum value is \( f(\pm 1,0) = 1 \).
Find the extreme values of the function \( f(x, y) = x^2 + 2y^2 \) on the disk \( x^2 + y^2 \leq 1 \).

[Sol]:

(1) We first look for possible local extremas of \( f(x, y) = x^2 + 2y^2 \) on the disk \( x^2 + y^2 < 1 \).

The critical number is

\[
\begin{cases}
  f_x = 2x = 0 \\
  f_y = 4y = 0 
\end{cases} \Rightarrow x = 0, y = 0
\]

and \( f(0,0) = 0 \).

(2) Compare this value with the extreme values on the boundary from Example 2:

\( f(0,\pm 1) = 2, \ f(\pm 1,0) = 1 \).

The maximum value of \( f \) on the disk \( x^2 + y^2 \leq 1 \) is \( f(0,\pm 1) = 2 \), and the minimum value is \( f(0,0) = 0 \).
[Ex4] Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3,1,-1)$.

[Sol]:

The distance of any point $(x, y, z)$ on the sphere to the point $(3,1,-1)$ is $d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}$.

Let $f(x, y, z) = (x-3)^2 + (y-1)^2 + (z+1)^2$. Then we want to maximize or minimize $f(x, y, z)$ subject to the constraint $g(x, y, z) = x^2 + y^2 + z^2 = 4$.

According to the Method of Lagrange Multiplier, we have to solve

$$
\begin{cases}
V_x = \lambda g_x \\
V_y = \lambda g_y \\
V_z = \lambda g_z \\
x^2 + y^2 + z^2 = 4
\end{cases} \Rightarrow \begin{cases}
2(x-3) = 2x\lambda \\
2(y-1) = 2y\lambda \\
2(z+1) = 2z\lambda \\
x^2 + y^2 + z^2 = 4
\end{cases} \Rightarrow \lambda = 1 \pm \frac{\sqrt{11}}{2}
$$

Since $f\left(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}}\right) < f\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$, we know that the closest point is $(\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, -\frac{2}{\sqrt{11}})$ and the farthest point is $\left(-\frac{6}{\sqrt{11}}, -\frac{2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right)$. 


Two Constraints (二個限制式)

Method of Lagrange Multipliers  
To find the maximum and minimum values of $f(x, y, z)$ subject to two constraints \( g(x, y, z) = k \) and \( h(x, y, z) = c \).

(a) Find all values of $x, y, z, \lambda$ and $\mu$ such that
\[
\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)
\]
\[
g(x, y, z) = k \quad \text{and} \quad h(x, y, z) = c
\]

(b) Evaluate $f$ at all the points $(x, y, z)$ that result from step (a). The largest of these values is the maximum value of $f$; the smallest is the minimum value of $f$.

Note: The equation  \( \nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z) \) is equivalent to
\[
f_x = \lambda g_x + \mu h_x, \quad f_y = \lambda g_y + \mu h_y, \quad \text{and} \quad f_z = \lambda g_z + \mu h_z
\]
[Ex5] Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

[Sol]:

We want to maximize $f(x, y)$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$.

According to the Method of Lagrange Multiplier with two constraints:

\[
\begin{align*}
\nabla f &= \lambda \nabla g + \mu \nabla h \\

\begin{cases}
    g(x, y, z) &= 1 \\
    h(x, y, z) &= 1
\end{cases}
\Rightarrow
\begin{cases}
    f_x &= \lambda g_x + \mu h_x \\
    f_y &= \lambda g_y + \mu h_y \\
    f_z &= \lambda g_z + \mu h_z
\end{cases}
\Rightarrow
\begin{cases}
    1 &= \lambda + 2x\mu \\
    2 &= -\lambda + 2y\mu \\
    3 &= \lambda \\
    x - y + z &= 1 \\
    x^2 + y^2 &= 1
\end{cases}
\Rightarrow
\begin{cases}
    \lambda = 3 \\
    \mu = \pm \frac{\sqrt{29}}{2}
\end{cases}
\Rightarrow
(x, y, z) = \left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) \text{ or } \left(-\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right)
\]

Since $f\left(-\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}, 1 + \frac{7}{\sqrt{29}}\right) = 3 + \sqrt{29} > f\left(-\frac{2}{\sqrt{29}}, -\frac{5}{\sqrt{29}}, 1 - \frac{7}{\sqrt{29}}\right) = 3 - \sqrt{29}$,

the maximum value of $f$ on the given curve is $3 + \sqrt{29}$.