3.1 Derivatives of Polynomials and Exponential Functions

**Differentiation Rules**

1. \( \frac{d}{dx} (c) = 0 \) (\( c \) is a constant)

2. The Power Rule
\( \frac{d}{dx} (x^n) = nx^{n-1} \) (\( n \): any real number)

3. The Constant Multiple Rule
\( \frac{d}{dx}[c \cdot f(x)] = c \frac{d}{dx} f(x) \)

4. The Sum Rule
\( \frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx} f(x) + \frac{d}{dx} g(x) \)

5. The Difference Rule
\( \frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx} f(x) - \frac{d}{dx} g(x) \)

**Examples**

\[ \frac{d}{dx} (5) = 0, \quad \frac{d}{dx} (15) = 0, \quad \frac{d}{dx} (-7) = 0 \]

\[ \frac{d}{dx} (x^3) = 3x^{3-1} = 3x^2 \]
\[ \frac{d}{dx} \left( \frac{1}{x^2} \right) = \frac{d}{dx} (x^{-2}) = -2x^{-3} \]
\[ \frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{1/2}) = \frac{1}{2}x^{-1/2} \]
\[ \frac{d}{dx} (5\sqrt{x}) = 5 \frac{d}{dx} (\sqrt{x}) = 5 \cdot \frac{1}{2}x^{-3/2} = \frac{5}{2}x^{-3/2} \]

\[ \frac{d}{dx} (x^3 + \frac{1}{x}) = \frac{d}{dx} x^3 + \frac{d}{dx} x^{-1} = 3x^2 - x^{-2} \]

\[ \frac{d}{dx} (x^m + 5x^2 - 1) = \frac{d}{dx} x^m + \frac{d}{dx} 5x^2 - \frac{d}{dx} 1 = mx^{m-1} + 5 \cdot 2x - 0 = mx^{m-1} + 10x \]
\[ \frac{d}{dx} \left( x^8 + 12x^5 - 4x^4 + 6x^3 - 6x + 5 \right) \]

\[ = \frac{d}{dx} x^8 + 12 \frac{d}{dx} x^5 - 4 \frac{d}{dx} x^4 + 10 \frac{d}{dx} x^3 - 6 \frac{d}{dx} (x) + \frac{d}{dx} 5 \]

\[ = 8x^7 + 12 \cdot 5x^4 - 4 \cdot 4x^3 + 10 \cdot 3x^2 - 6 + 0 \]

\[ = 8x^7 + 60x^4 - 16x^3 + 30x^2 - 6 \]

**6.** Find the points on the curve \( y = x^4 - 6x^2 + 4 \) where the tangent line is horizontal.

**601.** Let \( f(x) = x^4 - 6x^2 + 4 \). Then horizontal tangents occur where \( f'(x) = 0 \)

\[ f'(x) = 4x^3 - 12x \]

\[ f'(x) = 0 \iff 4x^3 - 12x = 0 \iff 4x(x^2 - 3) = 0 \iff x = 0, \ x = \pm \sqrt{3}. \]

Hence the given curve has horizontal tangents at \( (0, 4), (-\sqrt{3}, -5), (\sqrt{3}, -5) \) as the graph shown below.

Notice that the function \( f(x) \) is an even function and, as you can see, its graph is symmetric about the y-axis.
Def.

\( e \) is the number such that \( \lim_{h \to 0} \frac{e^h - 1}{h} = 1 \)

\[ f(x) = a^x \]

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h} = a^x \lim_{h \to 0} \frac{a^h - 1}{h} = a^x \lim_{h \to 0} \frac{a^h - a^0}{h} = a^x f'(0) \]

\[ f'(x) = a^x \cdot f(x) \]

for \( a = 2 \), \( f'(0) = \lim_{h \to 0} \frac{2^h - 1}{h} \approx 0.69 \)

for \( a = 3 \), \( f'(0) = \lim_{h \to 0} \frac{3^h - 1}{h} \approx 1.10 \)

So \( \frac{d}{dx} 2^x \approx (0.69) 2^x \) and \( \frac{d}{dx} 3^x \approx (1.10) 3^x \)

Therefore \( 2 < e < 3 \) and

(6) \( \frac{d}{dx} e^x = e^x \)
If \( f(x) = e^x - x \), find \( f' \). Compare the graphs of \( f \) and \( f' \).

\[
\begin{align*}
\frac{d}{dx}(e^x - x) &= \frac{d}{dx}(e^x) - \frac{d}{dx}(x) \\
&= e^x - 1
\end{align*}
\]

\( f \) has a horizontal tangent when \( x = 0 \) \( \iff \) \( f'(0) = 0 \)

\( f'(x) > 0 \) when \( x > 0 \) \( \implies \) \( f \) is increasing

\( f'(x) < 0 \) when \( x < 0 \) \( \implies \) \( f \) is decreasing.

At what point on the curve \( y = e^x \) is the tangent line parallel to the line \( y = 2x \)?

Suppose the point in question is \((a, e^a)\).

Then the slope of the tangent line at that point is

\[
y' \big|_{x=a} = e^x \big|_{x=a} = e^a
\]

Since the tangent line is parallel to the line \( y = 2x \), we have \( e^a = 2 \) (Parallel lines have the same slope)

\( \implies a = \ln 2 \)

Therefore the required point is \((\ln 2, e^{\ln 2}) = (\ln 2, 2)\).
3.2 The Product and Quotient Rules

**The Product Rule** \[ \frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}g(x) + g(x)\frac{d}{dx}f(x) \]

or \[ (f \cdot g)' = f \cdot (g') + g \cdot (f') \]

**Ex 1** Find \( f'(x) \) if \( f(x) = xe^x \)

**Sol** \[ f'(x) = \frac{d}{dx}(xe^x) = x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(x) = x e^x + e^x \cdot 1 = (x+1) e^x \]

**Ex 2** Differentiate the function \( f(t) = \sqrt{1-t} \)

**Sol 1** \[ f'(t) = \frac{d}{dt}(\sqrt{1-t}) = \sqrt{1-t} \cdot \frac{d}{dt}(1-t) + (1-t) \cdot \frac{d}{dt}(\sqrt{1-t}) = \sqrt{1-t} \cdot (-1) + (1-t) \cdot \frac{1}{2\sqrt{1-t}} = \frac{1-3t}{2\sqrt{1-t}} \]

**Sol 2** \[ f(t) = \sqrt{1-t} = t^{-\frac{1}{2}} - t^{\frac{3}{2}} \quad \therefore \quad f'(t) = \frac{3}{2} t^{-\frac{3}{2}} - \frac{1}{2} t^{-\frac{1}{2}} \]

**Ex 3** If \( f(x) = \sqrt{x} g(x) \), where \( g(4) = 2 \) and \( g'(4) = 3 \), find \( f'(4) \)

**Sol** \[ f'(x) = \frac{d}{dx}[\sqrt{x} g(x)] = \sqrt{x} \cdot \frac{d}{dx}g(x) + g(x) \cdot \frac{d}{dx}\sqrt{x} = \sqrt{x} \cdot g(x) + g(x) \cdot \frac{1}{2\sqrt{x}} \]

\[ \Rightarrow f'(4) = \sqrt{4} \cdot g(4) + g(4) \cdot \frac{1}{2\sqrt{4}} = 2 \cdot 2 + 2 \cdot \frac{1}{4} = 6 \frac{1}{2} \]
The Quotient Rule \[ \frac{d}{dx} \frac{f(x)}{g(x)} = \frac{g(x) \frac{d}{dx} f(x) - f(x) \frac{d}{dx} g(x)}{[g(x)]^2} \]

or \( \left( \frac{f}{g} \right)' = \frac{g \cdot f' - f \cdot g'}{g^2} \)

5] Let \( y = \frac{x^2 + x - 2}{x^2 + 6} \), find \( y' \).

\[
y' = \left( \frac{x^2 + 6}{x^3 + 6} \right) \cdot \frac{d}{dx} (x^2 + x - 2) - (x^3 + x - 2) \cdot \frac{d}{dx} (x^2 + 6) \quad \frac{(x^3 + 6) \cdot (x^2 + 1) - (x^3 + x - 2) \cdot 3x^2}{(x^3 + 6)^2}
\]

6] Find the eq. of the tangent line to the curve \( y = \frac{e^x}{1 + x^2} \) at the point \( (1, \frac{e}{2}) \).

\[
\frac{dy}{dx} = \frac{(1+x^2) \frac{d}{dx} (e^x) - e^x \frac{d}{dx} (1+x^2)}{(1+x^2)^2} = \frac{(1+x^2) \cdot e^x - e^x \cdot 2x}{(1+x^2)^2} = \frac{e^x(1-x)^2}{(1+x^2)^2}
\]

So the slope of the tangent line is \( \frac{dy}{dx} \bigg|_{x=1} = 0 \)

Therefore the eq. of the tangent line is \( y = \frac{e}{2} = 0 \cdot (x-1) \) or \( y = \frac{e}{2} \)

which is a horizontal line.
The Product Rule

\[
(f \cdot g \cdot h)' = f' \cdot g \cdot h + f \cdot g' \cdot h + f \cdot g \cdot h'
\]

\[
= f \cdot g \cdot h \left( \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h} \right)
\]

1] \( y = (2x+1)(3x+1)(4x+1). \) Find \( y' \)

\[\text{Sol:} \quad y' = \left[ \frac{d}{dx}(2x+1) \right] \cdot (3x+1) \cdot (4x+1) + (2x+1) \left[ \frac{d}{dx}(3x+1) \right] \cdot (4x+1) + (2x+1)(3x+1) \left[ \frac{d}{dx}(4x+1) \right]
\]

\[= (2)(3x+1)(4x+1) + (2x+1)(3)(4x+1) + (2x+1)(3x+1)(4)
\]

\[\text{or} \quad y' = (2x+1)(3x+1)(4x+1) \left[ \frac{(2x+1)'}{2x+1} + \frac{(3x+1)'}{3x+1} + \frac{(4x+1)'}{4x+1} \right]
\]

\[= (2x+1)(3x+1)(4x+1) \left[ \frac{2}{2x+1} + \frac{3}{3x+1} + \frac{4}{4x+1} \right]
\]

2] \( y = (x+1)(x^2+1)(x^3+1)(x^4+1) \)

\[\text{Sol:} \quad y' = \left[ \frac{d}{dx}(x+1) \right] (x^2+1)(x^3+1)(x^4+1) + (x+1) \left[ \frac{d}{dx}(x^2+1) \right] (x^3+1)(x^4+1) + (x+1)(x^2+1) \left[ \frac{d}{dx}(x^3+1) \right] (x^4+1) + (x+1)(x^2+1)(x^3+1) \left[ \frac{d}{dx}(x^4+1) \right]
\]

\[= (x+1)\left( x^2+1 \right) \left( x^3+1 \right) + (x+1) \cdot 2x \left( x^2+1 \right) \left( x^3+1 \right) + (x+1) \left( x^2+1 \right) \cdot 3x^2 \left( x^4+1 \right) + (x+1)(x^2+1)(x^3+1) \cdot 4x^3
\]

\[\text{or} \quad y' = (x+1)(x^2+1)(x^3+1)(x^4+1) \left[ \frac{(x+1)'}{x+1} + \frac{(x^2+1)'}{x^2+1} + \frac{(x^3+1)'}{x^3+1} + \frac{(x^4+1)'}{x^4+1} \right]
\]

\[= (x+1)(x^2+1)(x^3+1)(x^4+1) \left[ \frac{1}{x+1} + \frac{2x}{x^2+1} + \frac{3x^2}{x^3+1} + \frac{4x^3}{x^4+1} \right]
\]
3.4 Derivatives of Trigonometric Functions

\[ y = \sin x = f(x) \]

\[ y = f'(x) \]

\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} = \lim_{h \to 0} \frac{\cos \left( x + \frac{h}{2} \right) \sin \left( \frac{h}{2} \right)}{h} \\
= \lim_{h \to 0} \cos \left( x + \frac{h}{2} \right) \cdot \frac{\sin \left( \frac{h}{2} \right)}{\frac{h}{2}} = \lim_{h \to 0} \cos \left( x + \frac{h}{2} \right) \cdot \lim_{h \to 0} \frac{\sin \left( \frac{h}{2} \right)}{\frac{h}{2}} = \cos x \cdot 1 = \cos x \\
\]

**Thm**

\[ \lim_{x \to 0} \frac{\sin x}{x} = 1 \] (see next page)
[Proof]:

(i) when $0 < x < \frac{\pi}{2}$

$\triangle AOP < \triangle OAB < \triangle OAB$  

ie. $\frac{1}{2} \cdot 1 \cdot \sin x < \frac{1}{2} \cdot 1 \cdot x < \frac{1}{2} \cdot 1 \cdot \tan x$

$\Rightarrow \sin x < x < \tan x$

$\Rightarrow 1 < \frac{x}{\sin x} < \frac{\tan x}{\sin x} = \frac{1}{\cos x} \quad (\because \sin x > 0)$

$\Rightarrow 1 > \frac{\sin x}{x} > \cos x \quad \text{for } 0 < x < \frac{\pi}{2}$

Since $\lim_{x \to 0^+} 1 = \lim_{x \to 0^+} \cos x = 1$

By squeeze thm, we have $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$

(ii) when $-\frac{\pi}{2} < x < 0$

Since $0 < (-x) < \frac{\pi}{2}$, by (i) we have $1 > \frac{\sin(-x)}{-x} > \cos(-x)$

ie. $1 > \frac{-\sin x}{-x} > \cos x$  ie. $1 > \frac{\sin x}{x} > \cos x$ for $-\frac{\pi}{2} < x < 0$

With $\lim_{x \to 0^-} 1 = \lim_{x \to 0^-} \cos x = 1$, we have $\lim_{x \to 0^-} \frac{\sin x}{x} = 1$

(iii) By (i) and (ii), $\lim_{x \to 0^+} \frac{\sin x}{x} = \lim_{x \to 0^-} \frac{\sin x}{x} = 1$

Therefore, $\lim_{x \to 0} \frac{\sin x}{x} = 1$
Derivatives of Trigonometric Functions:

\[
\frac{d}{dx} \sin x = \cos x \\
\frac{d}{dx} \cos x = -\sin x \\
\frac{d}{dx} \tan x = \sec^2 x \\
\frac{d}{dx} \cot x = -\csc^2 x \\
\frac{d}{dx} \sec x = \sec x \tan x \\
\frac{d}{dx} \csc x = -\csc x \cot x
\]

Proof of \( \frac{d}{dx} \tan x = \sec^2 x \):

\[
\frac{d}{dx} (\tan x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = \frac{\frac{d}{dx} \sin x \cos x - \sin x \frac{d}{dx} \cos x}{(\cos x)^2} = \frac{\cos x \cdot \cos x - \sin x (-\sin x)}{\cos x} \\
= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x
\]
1. Differentiate $y = x^2 \sin x$

$$\frac{dy}{dx} = (d\frac{dx}{dx}) \cdot \sin x + x^2 \cdot \frac{d}{dx} \sin x$$

$$= 2x \sin x + x^2 \cos x$$

2. Differentiate $f(x) = \frac{\sec x}{1 + \tan x}$. For what values of $x$ does the graph of $f$ have a horizontal tangent?

At $x = \pi n + \frac{\pi}{4}$, the curve has a horizontal tangent.
3. An object at the end of a vertical spring is stretched 4 cm beyond its rest position and released at time $t=0$. (Note that the downward direction is positive). Its position at time $t$ is $s(t) = 4 \cos t$. Find the velocity at time $t$ and use it to analyze the motion of the object.

$$u = \frac{ds}{dt} = -4 \sin t$$

$$|v| = 4|\sin t|$$

4. Find $\lim_{x \to 0} \frac{\sin 7x}{4x}$

5. Calculate $\lim_{x \to 0} x \cot x$
The Chain Rule

If \( f \) and \( g \) are both differentiable, then \( F = f \circ g \) is differentiable and \( F' \) is given by

\[
F'(x) = f'(g(x)) \cdot g'(x)
\]

or

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}
\]

if \( y = f(u) \) and \( u = g(x) \)

---

**Example 1**

Find \( F'(x) \) if \( F(x) = \sqrt{x^2 + 1} \)

**Solution**:

\[
F'(x) = \frac{d}{dx} (x^2 + 1)^{\frac{1}{2}} = \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot 2x = \frac{1}{2 \sqrt{x^2 + 1}}
\]

or:

Let \( y = f(u) = u^{\frac{1}{2}}, \ u = g(x) = x^2 + 1 \). then \( F(x) = f(g(x)) \)

By the Chain Rule, \( F'(x) = f'(g(x)) \cdot g'(x) = \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot 2x \)

or

\[
\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{2} u^{-\frac{1}{2}} \cdot 2x = \frac{1}{2} (x^2 + 1)^{-\frac{1}{2}} \cdot 2x
\]
(2) Differentiate \( y = \sin(x^2) \) and \( y = \sin^2x \)

**Solution:**

(a) \[
\frac{dy}{dx} = \frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot \frac{d}{dx} (x^2) = \cos(x^2) \cdot 2x = 2x \cos(x^2)
\]

(b) \[
\frac{dy}{dx} = \frac{d}{dx} \sin^2x = \frac{d}{dx} (\sin x)^2 = 2(\sin x) \cdot \frac{d}{dx} (\sin x) = 2 \sin x \cdot \cos x
\]

[i] Suppose that \( f(x) = x^2 \) and \( y = f\left(\frac{2x-1}{x+1}\right) \). Find \( \frac{dy}{dx} \)

\[
\frac{dy}{dx} = \frac{d}{dx} f\left(\frac{2x-1}{x+1}\right) = f'\left(\frac{2x-1}{x+1}\right) \cdot \frac{d}{dx}\left(\frac{2x-1}{x+1}\right)
\]

\[
= \left(\frac{2x-1}{x+1}\right)^2 \cdot \frac{2 \cdot (x+1) - (2x-1) \cdot 1}{(x+1)^2} = \frac{3(2x-1)^2}{(x+1)^4}
\]

[i] If \( \frac{d}{dx} f(x^2) = x^3 \). Find \( f'(3) \)

\[\text{By the Chain Rule, } \frac{d}{dx} f(x^2) = f'(x^2) \cdot 2x \text{ i.e. } x^3 = f'(x^2) \cdot 2x \]

Therefore, \[f'(x^2) = \frac{x^3}{2x} = \frac{x^2}{2}\]

Hence, \[f''(3) = \frac{3}{2}\].
Differentiate \( y = e^{\sin x} \)

\[
\frac{dy}{dx} = \frac{d}{dx} e^{\sin x} = e^{\sin x} \cdot \frac{d}{dx} (\sin x) = e^{\sin x} \cdot \cos x
\]

Observe the last example, we have the following rule.

\[
\frac{d}{dx} e^u = e^u \cdot u', \text{ if } u \text{ is differentiable.}
\]

Using this rule, we can deduce the formulas

1. \( \frac{d}{dx} (a^x) = a^x \ln a \)
2. \( \frac{d}{dx} (a^u) = a^u \ln a \cdot u' \)

**Proof**

1. \( \frac{d}{dx} (a^x) = \frac{d}{dx} (e^{x \ln a}) = \frac{d}{dx} (e^{x \ln a}) = e^{x \ln a} \cdot \frac{d}{dx}(\ln a \cdot x) = e^{x \ln a} \cdot \ln a = a^x \ln a \)

2. By the Chain Rule and (1).

\( \frac{d}{dx} (a^u) = a^u \ln a \cdot u' \)
Find \( y' \):

1. \( y = e^{\sec^2 \theta} \)
   \[ y' = \frac{dy}{d\theta} = \frac{d}{d\theta} e^{\sec^2 \theta} = e^{\sec^2 \theta} \cdot \frac{d}{d\theta} (\sec^2 \theta) = e^{\sec^2 \theta} \cdot \sec(2\theta) \tan(\theta) \cdot 2 \]

2. \( y = 5^x \)
   \[ y' = \frac{d}{dx} (5^x) = 5^x \cdot \ln 5 \]

3. \( y = 10^{\cos x} \)
   \[ y' = \frac{d}{dx} (10^{\cos x}) = 10^{\cos x} \cdot \ln 10 \cdot (-\sin x) \]

If \( f(x) = \sin (\cos (\tan x)) \), find \( f'(x) \).

\[ f'(x) = \frac{d}{dx} \sin (\cos (\tan x)) \]
\[ = \sin (\cos (\tan x)) \cdot \frac{d}{dx} (\cos (\tan x)) \]
\[ = \sin (\cos (\tan x)) \cdot (-\sin (\tan x)) \cdot \frac{d}{dx} (\tan x) \]
\[ = -\sin (\tan x) \cdot \sin (\cos (\tan x)) \cdot \sec^2 x \]
The Extended Power Rule.

\[ \frac{d}{dx}(u^n) = n \cdot u^{n-1} \cdot \frac{du}{dx} \quad \text{or} \quad \frac{d}{dx}(f(x))^n = n \cdot (f(x))^{n-1} \cdot f'(x) \]

\[ x^3 \] Differentiate \( y = (x^3 - 1)^{100} \)

\[ \text{Sol1:} \quad \frac{dy}{dx} = \frac{d}{dx}[(x^3 - 1)^{100}] = 100 \cdot (x^3 - 1)^{99} \cdot \frac{d}{dx}(x^3 - 1) = 100 \cdot (x^3 - 1)^{99} \cdot 3x^2 \]

\[ x^4 \] Find \( f(x) \) if \( f(x) = \frac{1}{\sqrt[3]{x^2 + x + 1}} \)

\[ \text{Sol1:} \quad f'(x) = \frac{d}{dx}\left(\frac{1}{\sqrt[3]{x^2 + x + 1}}\right) = \frac{d}{dx}(x^2 + x + 1)^{-\frac{1}{3}} = -\frac{1}{3} \cdot (x^2 + x + 1)^{-\frac{4}{3}} \cdot \frac{d}{dx}(x^2 + x + 1) = \frac{-(2x+1)}{3(3x^2 + 3x + 1)^{\frac{4}{3}}} \]

\[ x^5 \] Find the derivative of the function \( g(x) = \left(\frac{t-2}{2t+1}\right)^9 \)

\[ \text{Sol1:} \quad g'(x) = \frac{d}{dx}\left(\frac{t-2}{2t+1}\right)^9 = 9 \cdot \left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{d}{dx}\left(\frac{t-2}{2t+1}\right) = 9 \cdot \left(\frac{t-2}{2t+1}\right)^8 \cdot \frac{1 \cdot (2t+1) - (t-2) \cdot 2}{(2t+1)^2} \]

\[ = \frac{45(t-2)^8}{(2t+1)^{10}} \]
Differentiate \( y = (2x+1)^5 (x^3 - x+1)^4 \)

\[
\frac{dy}{dx} = \left[ \frac{d}{dx} (2x+1)^5 \right] \cdot (x^3 - x+1)^4 + (2x+1)^5 \left[ \frac{d}{dx} (x^3 - x+1)^4 \right]
\]

\[
= \left[ 5 \cdot (2x+1)^4 \cdot \frac{d}{dx}(2x+1) \right] \cdot (x^3 - x+1)^4 + (2x+1)^5 \left[ 4 \cdot (x^3 - x+1)^3 \cdot \frac{d}{dx} (x^3 - x+1) \right]
\]

\[
= 5 (2x+1)^4 \cdot 2 \cdot (x^3 - x+1)^4 + 4 (2x+1)^5 \cdot (x^3 - x+1)^3 \cdot (3x^2 - 1)
\]

\[
= 10 (2x+1)^4 \cdot (x^3 - x+1)^4 + 4 (2x+1)^5 \cdot (x^3 - x+1)^3 \cdot (3x^2 - 1)
\]
3.6 Implicit Differentiation

The function \( f(x) = \frac{1}{x} \) can be expressed explicitly as \( f(x) = \frac{1}{x} \) or \( y = \frac{1}{x} \), or expressed implicitly as \( xy = 1 \).

For example,

\( y = \sqrt{x^2 + 1} \) or \( y = x \sin x \) are functions defined explicitly.

\( x^2 + y^2 = 5 \) or \( x^2 + 5 \sin(x + y) = y^2 \) are functions defined implicitly. (They are just relations between \( x \) and \( y \)).
Two of the functions determined by the implicit eq. \( x^2 + y^2 = 5 \) are
\[
f(x) = \sqrt{5 - x^2} \quad \text{and} \quad g(x) = -\sqrt{5 - x^2}.
\]

\textbf{x1] (a)}: If \( x^2 + y^2 = 5 \), find \( \frac{dy}{dx} \)

\( \text{(b)} \): Find an eq. of the tangent line to the circle \( x^2 + y^2 = 5 \) at the point \((3,4)\)

\textbf{x1] (a)}: Differentiate both sides of the eq. \( x^2 + y^2 = 5 \):
\[
\frac{d}{dx} (x^2 + y^2) = \frac{d}{dx} (5)
\Rightarrow 2x + 2y \cdot \frac{dy}{dx} = 0
\Rightarrow \frac{dy}{dx} = -\frac{x}{y}
\]

If you solve the eq. for \( y \) as explicit function of \( x \):
\[
x^2 + y^2 = 5 \Rightarrow y = \sqrt{5 - x^2} \quad \text{or} \quad y = -\sqrt{5 - x^2}
\]

(i) \( \text{if} \quad y = \sqrt{5 - x^2} \), then \( y' = \frac{1}{2} (5 - x^2)^{-\frac{1}{2}} (-2x) = -\frac{x}{\sqrt{5 - x^2}} = -\frac{x}{y} \)

(ii) \( \text{if} \; y = -\sqrt{5 - x^2} \), then \( y' = -\frac{1}{2} (5 - x^2)^{-\frac{1}{2}} (-2x) = -\frac{x}{\sqrt{5 - x^2}} = -\frac{x}{y} \)
(b) the slope of the tangent line to the circle at (3, 4) 
\[ \frac{dy}{dx} \bigg|_{(3, 4)} = \frac{dy}{dx} \bigg|_{x=3, y=4} = -\frac{x}{y} \bigg|_{(3, 4)} = -\frac{3}{4} \]

Therefore, the eq. of the tangent line is
\[ y - 4 = -\frac{3}{4} (x - 3) \]

OR:
\[ x^2 + y^2 = 25 \Rightarrow y = \pm \sqrt{25 - x^2} \]

The point (3, 4) lies on the upper semicircle \( y = \sqrt{25 - x^2} \)

Let \( f(x) = \sqrt{25 - x^2} \)
\[ \Rightarrow f'(x) = \frac{1}{2} (25 - x^2)^{-\frac{1}{2}} (-2x) = -\frac{x}{\sqrt{25 - x^2}} \]

The slope of the tangent line at (3, 4) = \( f'(3) = -\frac{3}{4} \)

Therefore, the eq. of the tangent line is
\[ y - 4 = -\frac{3}{4} (x - 3) \]

※ The expression \( \frac{dy}{dx} = -\frac{x}{y} \) gives the derivative in terms of both \( x \) and \( y \).
It is correct no matter which function is determined by the given eq.
(2) (a) Find \( y' \) if \( x^3 + y^3 = 6xy \)

(b) Find the tangent to the folium of Descartes \( x^3 + y^3 = 6xy \) at the point \((3, 3)\)

(c) At what points on the curve is the tangent line horizontal?

\[
\frac{d}{dx} (x^3 + y^3) = \frac{d}{dx} (6xy)
\]

\[
3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6x \frac{dy}{dx} \Rightarrow (3y^2 - 6x) \frac{dy}{dx} = 6y - 3x^2 \Rightarrow \frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}
\]

(b) The slope of the tangent at \((3, 3)\) = \( \left. \frac{dy}{dx} \right|_{(3, 3)} = -1\)

Therefore, the eq. of the tangent line at \((3, 3)\) is

\[y - 3 = -1(x - 3) \text{ or } x + y = 6\]

(c) horizontal tangent \(\Leftrightarrow y' = 0\)

\[6y - 3x^2 = 0 \Leftrightarrow y = \frac{1}{2}x^2\]

Sub. \(y = \frac{1}{2}x^2\) into the eq., we have

\[x^3 + \left(\frac{1}{2}x^2\right)^3 = 6x \left(\frac{1}{2}x^2\right)\]

\[\Rightarrow x^6 = 16x^3 \Rightarrow x = 0 \text{ or } x = 2^{\frac{3}{2}}\]

When \(x = 0\), \(y = 0\); when \(x = 2^{\frac{3}{2}}\), \(y = \frac{1}{2}(2^{\frac{3}{2}})^2 = 2^{\frac{5}{2}}\)

Thus, the tangent is horizontal at \((0, 0)\) and at \((2^{\frac{3}{2}}, 2^{\frac{5}{2}})\)
The three roots of the cubic equation \( x^3 + y^3 = 6xy \) are

\[
y = f(x) = \frac{1}{2} \left[ -\frac{1}{3} x^3 + \sqrt{\frac{4}{27} x^6 - 8x^3} \right] + \frac{1}{2} \left[ -\frac{1}{3} x^3 - \sqrt{\frac{4}{27} x^6 - 8x^3} \right]
\]

\[
y = \frac{1}{2} \left[ -f(x) \pm \sqrt{3} \left( \frac{1}{3} x^3 + \sqrt{\frac{4}{27} x^6 - 8x^3} - \frac{1}{3} x^3 - \sqrt{\frac{4}{27} x^6 - 8x^3} \right) \right]
\]
Find $y'$ if $\sin(x+y) = y^2 \cos x$

[Sol]:
\[
\frac{d}{dx} \sin(x+y) = \frac{d}{dx}(y^2 \cos x)
\]

\[
\Rightarrow \cos(x+y) \cdot (1 + \frac{dy}{dx}) = 2y \frac{dy}{dx} \cos x + y^2 \cdot (-\sin x)
\]

\[
\Rightarrow (\cos(x+y) \cdot y \cos x) \frac{dy}{dx} = -y^2 \sin x - \cos(x+y)
\]

\[
\Rightarrow \frac{dy}{dx} = -\frac{y^2 \sin x + \cos(x+y)}{\cos(x+y) - y \cos x}
\]
Orthogonal Trajectories

1. Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular.

2. Two families of curves are **orthogonal trajectories** of each other, if every curve in one family is orthogonal to every curve in the other family.

The eq. \( xy = C \) \((C \neq 0)\) represents a family of hyperbolas. The eq. \( x^2 - y^2 = k \) \((k \neq 0)\) represents another family of hyperbolas with asymptotes \( y = \pm x \). Show that the two families are orthogonal trajectories of each other.

1:
\[
\frac{d}{dx}(xy) = \frac{d}{dx}(C) \Rightarrow 1 \cdot y + x \cdot \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -\frac{y}{x}
\]
\[
\frac{d}{dx}(x^2y^2) = \frac{d}{dx}(k) \Rightarrow 2x - 2y \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}
\]

Since \( \left(-\frac{y}{x}\right) \left(\frac{x}{y}\right) = -1 \), we know that at any point of intersection, the tangent lines of the curves from each family are perpendicular. That means the curves are orthogonal. Therefore the two families are orthogonal trajectories of each other.
Derivatives of Inverse Trigonometric Functions

\[ \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}}, \ -1 < x < 1 \]

\[ \frac{d}{dx}(\cos^{-1}x) = -\frac{1}{\sqrt{1-x^2}}, \ -1 < x < 1 \]

\[ \frac{d}{dx}(\tan^{-1}x) = \frac{1}{1+x^2}, \ -\infty < x < \infty \]

\[ \frac{d}{dx}(\cot^{-1}x) = -\frac{1}{1+x^2}, \ -\infty < x < \infty \]

\[ \frac{d}{dx}(\sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}}, \ x>1 \text{ or } x<-1 \]

\[ \frac{d}{dx}(\csc^{-1}x) = -\frac{1}{x\sqrt{x^2-1}}, \ x>1 \text{ or } x<1 \]

[The proof of \( \frac{d}{dx}(\sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \):]

Let \( y = \sin^{-1}x, \ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \)

\( \Leftrightarrow \sin y = x \)

\( \frac{d}{dx}(\sin y) = \frac{d}{dx}(x) \)

\( \cos y \cdot \frac{dy}{dx} = 1 \)

\( \frac{dy}{dx} = \frac{1}{\cos y} \)

\( \Rightarrow \frac{1}{\sqrt{1-\sin^2 y}} \) (\( \because -\frac{\pi}{2} \leq y \leq \frac{\pi}{2} \))

\( = \frac{1}{\sqrt{1-x^2}} \) (\( \because \cos y \geq 0 \))
The proof of \( \frac{d}{dx} \sec^4 x = \frac{1}{x \sqrt{x^2 - 1}} \):

Let \( y = \sec^4 x \), \( 0 \leq y < \frac{\pi}{2} \), \( \pi \leq y < \frac{3\pi}{2} \)

\( \iff \) \( \sec y = x \)

\[ \frac{d}{dx} \sec y = \frac{d}{dx} x \]

\[ \implies \sec y \cdot \frac{dy}{dx} = 1 \]

\[ \Rightarrow \frac{dy}{dx} = \frac{1}{\sec y \cdot \tan y} \]

\[ = \frac{1}{\sec y \cdot \sqrt{x^2 - 1}} \quad (\because \sec y \cdot \tan y \geq 0) \]

$\therefore$ Differentiate (a) \( y = \frac{1}{\sin^{-1} x} \)

(b) \( f(x) = x \tan^{-1} x \)

B1:

(a) \( y = (\sin^{-1} x)^{-1} \Rightarrow y' = - (\sin^{-1} x)^3 \cdot \frac{d}{dx} \sin^{-1} x = - \left( \frac{1}{\sin^{-1} x} \right)^2 \cdot \frac{1}{\sqrt{1-x^2}} \)

(b) \( f'(x) = (\frac{d}{dx} x) \cdot \tan^{-1} x + x \cdot \frac{d}{dx} \tan^{-1} x = 1 \cdot \tan^{-1} x + x \cdot \frac{1}{1+(1/x)^2} \cdot \frac{d}{dx} x = \tan^{-1} x + x \cdot \frac{1}{1+x^2} \cdot \frac{1}{2x} \)
3.7 Higher Derivatives

\[ y = f(x) \]

the first derivative: \[ y' = \frac{d}{dx} y = \frac{dy}{dx} = f'(x) = \frac{d}{dx} f(x) = \frac{d^2 f(x)}{dx^2} = Df(x) \]

the second derivative: \[ y'' = \frac{d}{dx} (\frac{dy}{dx}) = \frac{d^2 y}{dx^2} = D^2 y \]

\[ f''(x) = \frac{d}{dx} (\frac{d}{dx} f(x)) = \frac{d^2}{dx^2} f(x) = \frac{d^3 f(x)}{dx^3} = D^3 f(x) \]

the third derivative: \[ y''' = \frac{d}{dx} (\frac{d^2 y}{dx^2}) = \frac{d^3 y}{dx^3} = D^3 y \]

\[ f'''(x) = \frac{d}{dx} (\frac{d^2}{dx^2} f(x)) = \frac{d^3}{dx^3} f(x) = \frac{d^4 f(x)}{dx^4} = D^4 f(x) \]

the fourth derivative: \[ y^{(4)} = \frac{d^4}{dx^4} y = \frac{d^4 y}{dx^4} = f^{(4)}(x) = \frac{d^4}{dx^4} f(x) = \frac{d^5 f(x)}{dx^5} = D^5 f(x) \]

\[ \vdots \]

the n-th derivative: \[ y^{(n)} = \frac{d^n}{dx^n} y = \frac{d^n y}{dx^n} = f^{(n)}(x) = \frac{d^n}{dx^n} f(x) = \frac{d^{n+1} f(x)}{dx^{n+1}} = D^n f(x) \]

\[ S(t) \rightarrow S'(t) = v(t) \rightarrow S''(t) = a(t) \]

position function velocity function acceleration function
3.7-2

3. If \( y = x^3 - 6x^2 - 5x + 3 \), find \( y^{(n)} \)

Solution:
\[
\begin{align*}
y' &= 3x^2 - 12x - 5 \\
y'' &= 6x - 12 \\
y''' &= 6 \\
y^{(4)} &= 0 \\
y^{(n)} &= 0 \quad \text{for } n \geq 4
\end{align*}
\]

4. If \( f(x) = \frac{1}{x} \), find \( f^{(n)}(x) \)

Solution:
\[
\begin{align*}
f(x) &= \frac{1}{x} = x^{-1} \\
f'(x) &= -x^{-2} \\
f''(x) &= (-1)(-2)x^{-3} \\
f'''(x) &= (-1)(-2)(-3)x^{-4} \\
f^{(4)}(x) &= (-1)(-2)(-3)(-4)x^{-5} \\
&\vdots \\
f^{(n)}(x) &= (-1)(-2)(-3)\ldots(-n)x^{-(n+1)} \quad \text{for } n \geq 1
\end{align*}
\]

Therefore,
\[
f^{(100)}(3) = \frac{(-1)^{100} \cdot 100!}{3^{101}}
\]

\[
= \frac{100!}{3^{101}}
\]
Find $D^{n_4} \cos x$.

0.] $D \cos x = -\sin x$
$D^2 \cos x = -\cos x$
$D^3 \cos x = \sin x$
$D^4 \cos x = \cos x$

$\Rightarrow \left\{\begin{array}{l}
D^{4k+1} \cos x = -\sin x \\
D^{4k+2} \cos x = -\cos x \\
D^{4k+3} \cos x = \sin x \\
D^{4k} \cos x = \cos x \\
\end{array}\right.$
Therefore $D^7 \cos x = \sin x$

x5] Find $y''$, if $x^4 + y^4 = 16$

[Sol]:
(1) find $y'$: $\frac{d}{dx} (x^4 + y^4) = \frac{d}{dx} (16) \Rightarrow 4x^3 + 4y^3 \cdot y' = 0 \Rightarrow y' = -\frac{x^3}{y^3}$

(2) find $y''$: $y'' = \frac{d}{dx} (y') = \frac{d}{dx} (-\frac{x^3}{y^3}) = -\frac{3x^2 \cdot y^3 - x^3 \cdot 3y^2 \cdot y'}{(y^3)^2}$

$= -\frac{3x^2 y^3 - x^3 \cdot 3y^2 (-\frac{x^3}{y^3})}{y^6} = -\frac{3x^4 y^3 + 3 \cdot \frac{x^6}{y^6}}{y^6}$

$= -\frac{3x^3 y^6 + 3x^6}{y^7} = -\frac{3x^2 (y^4 + x^4)}{y^7} = -\frac{3x^2 \cdot 16}{y^7} = -\frac{48x^2}{y^7}$

(\because x^4 + y^4 = 16)
3.8 Derivatives of Logarithmic Functions

\[
\frac{d}{dx} \log_a x = \frac{1}{x \ln a}
\]

In particular,

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

[Proof] Let \( y = \log_a x \Rightarrow x = a^y \)

\[
\frac{d}{dx} (a^y) = \frac{d}{dx} x \Rightarrow a^y \cdot y' \ln a = 1 \Rightarrow y' = \frac{1}{a^y \ln a} = \frac{1}{x \ln a}
\]

[EX1] Differentiate \( y = \ln (x^2+1) \)

[Sol]: \[
\frac{dy}{dx} = \frac{1}{x^2+1} \cdot \frac{d}{dx} (x^2+1) = \frac{3x^2}{x^2+1}
\]

\[
\frac{d}{dx} \ln u = \frac{u'}{u}
\]

\[
\frac{d}{dx} \log_a u = \frac{u'}{u \ln a}
\]

\[
\frac{d}{dx} \ln |u| = \frac{u'}{u}
\]

\[
\frac{d}{dx} \log_a |u| = \frac{u'}{u \ln a}
\]

(proof: 
\(\text{ii) if } u > 0 \text{ then } \frac{d}{dx} \ln u = \frac{d}{dx} \ln u = \frac{u'}{u} \)
\(\text{iii) if } u < 0 \text{ then } \frac{d}{dx} \ln |u| = \frac{d}{dx} \ln (-u) = \frac{u'}{-u} = \frac{u'}{u} \)

by (i), (ii) we conclude that \(\frac{d}{dx} \ln |u| = \frac{u'}{u} \)
\[ f(x) = \ln(2x+1), \quad D_f = \{ x \mid x > -\frac{1}{2} \} \]

\[ g(x) = \ln|2x+1|, \quad D_g = \{ x \mid x > -\frac{1}{2} \} \]

\[
\begin{align*}
  f'(x) &= \frac{d}{dx} \ln(2x+1) = \frac{2}{2x+1}, \quad x > -\frac{1}{2} \\
  g'(x) &= \frac{d}{dx} \ln|2x+1| = \frac{2}{2x+1}, \quad x > -\frac{1}{2}
\end{align*}
\]

Therefore,

\[
\begin{align*}
  \frac{d}{dx} \ln|\cos x| &= \frac{-\sin x}{\cos x} = -\tan x, \quad \cos x \neq 0 \\
  \frac{d}{dx} \ln(\cos x) &= \frac{-\sin x}{\cos x} = -\tan x, \quad \cos x > 0
\end{align*}
\]
Examples:

1. \( \frac{d}{dx} \ln(|\sin x|) = \frac{\cos x}{\sin x} \)

2. \( \frac{d}{dx} \ln(\sin x) = \frac{\cos x}{\sin x} \)

3. \( \frac{d}{dx} \log_3(\sin x) = \frac{\cos x}{\sin x (\ln 3)} \)

4. \( \frac{d}{dx} \log_{10}(3x + \tan x) = \frac{3 + \sec^2 x}{(3x + \tan x)(\ln 10)} \)

5. \( \frac{d}{dx} \sqrt{\ln x} = \frac{1}{2} \frac{d}{dx} (\ln x)^{\frac{1}{2}} = \frac{1}{\sqrt{\ln x}} \cdot \frac{1}{x} \)

6. \( \frac{d}{dx} \ln \left( \frac{x+1}{\sqrt{x-2}} \right) = \frac{d}{dx} \left[ \ln(x+1) - \frac{1}{2} \ln(x-2) \right] = \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-2} \)

7. \( \frac{d}{dx} \ln(|\sec x + \tan x|) = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x \cdot \sec^2 x \)

8. \( \frac{d}{dx} \log_{\sqrt{7}}\left( \frac{x^2 + 3}{4x - 3} \right) = \frac{d}{dx} \left( \log_7(x^2 + 3) - \log_7(4x - 3) \right) = \frac{2x}{x^2 + 3 (\ln 7)} - \frac{4}{(4x - 3)(\ln 7)} \)
Logarithmic Differentiation

1. Take natural logarithms of both sides of an eq. \( y = f(x) \) and use the Laws of Logarithms to simplify.
2. Differentiate implicitly w.r.t. \( x \).
3. Solve the resulting eq. for \( y' \).

[Ex 7] Differentiate \( y = \frac{x^{\frac{3}{4}} \sqrt{x^2 + 1}}{(3x+2)^5} \)

[Sol] 1. \( \ln y = \ln \left( \frac{x^{\frac{3}{4}} \sqrt{x^2 + 1}}{(3x+2)^5} \right) \)

\[ \Rightarrow \ln y = \frac{3}{4} \ln x + \frac{1}{2} \ln (x^2 + 1) - 5 \ln (3x+2) \]

2. Differentiate both sides w.r.t. \( x \), we get

\[ \frac{y'}{y} = \frac{3}{4} \cdot \frac{1}{x} + \frac{1}{2} \cdot \frac{2x}{x^2 + 1} - 5 \cdot \frac{3}{3x+2} \]

3. Solve for \( y' \):

\[ y' = y \left[ \frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x+2} \right] = \frac{x^{\frac{3}{4}} \sqrt{x^2 + 1}}{(3x+2)^5} \left[ \frac{3}{4x} + \frac{x}{x^2 + 1} - \frac{15}{3x+2} \right] \]
EX8] Differentiate \( y = x^{\frac{1}{x}} \)

[Sol.]
\[
\ln y = \frac{1}{x} \cdot \ln x
\]
\[
\Rightarrow \ln y = \frac{1}{\sqrt{x}} \cdot \ln x
\]
Differentiate implicitly w.r.t. \( x \):
\[
\frac{y'}{y} = \frac{1}{2} \cdot x^{-\frac{3}{2}} \cdot \ln x + \frac{1}{2} \cdot \left( \frac{1}{\sqrt{x}} \right) - \frac{1}{x} = \frac{\ln x + 2}{2\sqrt{x}}
\]
\[
\Rightarrow y' = y \left( \frac{\ln x + 2}{2\sqrt{x}} \right) = x^{\frac{1}{x}} \left( \frac{\ln x + 2}{2\sqrt{x}} \right)
\]

EX9] Differentiate \( y = (\ln x)^{\sin x} \)

[Sol.]
\[
\ln y = \ln (\ln x)^{\sin x} = \sin x \cdot \ln (\ln x)
\]
\[
\Rightarrow \frac{y'}{y} = (\frac{d}{dx} \sin x) \cdot \ln (\ln x) + \sin x \cdot (\frac{d}{dx} \ln (\ln x))
\]
\[
\Rightarrow \frac{y'}{y} = \cos x \cdot \ln (\ln x) + \sin x \cdot \frac{1}{x} \cdot \frac{1}{\ln x}
\]
\[
\Rightarrow y' = y \cdot \left[ \cos x \cdot \ln (\ln x) + \sin x \cdot \frac{1}{x} \cdot \frac{1}{\ln x} \right]
\]
\[
= (\ln x)^{\sin x} \left[ \cos x \cdot \ln (\ln x) + \frac{\sin x}{x \cdot \ln x} \right]
\]
The number $e$ as a limit

We have shown that if $f(x) = \ln x$, then $f'(x) = \frac{1}{x}$. Thus, $f'(1) = 1$

By the definition of the derivative of $f$ at $x=1$, we have

$$f'(1) = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to 0} \frac{\ln(1+x) - \ln 1}{x} = \lim_{x \to 0} \frac{1}{x} \ln(1+x) = \lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}$$

Since $f'(1) = 1$, we get $\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}} = 1$

Therefore, $e = e^1 = e^{\lim_{x \to 0} \ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} e^{\ln(1+x)^{\frac{1}{x}}} = \lim_{x \to 0} (1+x)^{\frac{1}{x}}$

($e^x$ is cont. at $x=1$)

\[ e = \lim_{x \to 0} (1+x)^{\frac{1}{x}} \]

\[ = \lim_{n \to \infty} (1 + \frac{1}{n})^n \]

\[ \approx 2.7182818 \text{ (Correct to seven decimal places)} \]
3.9 Hyperbolic Functions

The hyperbolic sine
$$\sinh(x) = \frac{e^x - e^{-x}}{2}$$

The hyperbolic cosine
$$\cosh(x) = \frac{e^x + e^{-x}}{2}$$

The hyperbolic tangent
$$\tanh(x) = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

The hyperbolic cotangent
$$\coth(x) = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The hyperbolic secant
$$\text{sech}(x) = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

The hyperbolic cosecant
$$\text{csch}(x) = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$
Hyperbolic Identities

\[ \sinh(-x) = -\sinh x \]
\[ \cosh^2(x) - \sinh^2(x) = 1 \]
\[ 1 - \tanh^2(x) = \text{sech}^2 x \]
\[ \sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y \]
\[ \cosh(x+y) = \cosh x \cosh y - \sinh x \sinh y \]

Derivatives of Hyperbolic Functions

\[ \frac{d}{dx} (\sinh x) = \cosh x \]
\[ \frac{d}{dx} (\cosh x) = \sinh x \]
\[ \frac{d}{dx} (\tanh x) = \text{sech}^2 x \]
\[ \frac{d}{dx} (\coth x) = -\text{csch}^2 x \]
\[ \frac{d}{dx} (\text{sech} x) = -\text{sech} x \tanh x \]
\[ \frac{d}{dx} (\text{csch} x) = -\text{csch} x \coth x \]
Example 1: Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm³/s. How fast is the radius of the balloon increasing when the diameter is 0 cm?

We are given \( \frac{dV}{dt} = 100 \text{ cm}^3/\text{s} \) and we are asked to find \( \frac{dr}{dt} \bigg|_{r=25} \)

Since \( V = \frac{4}{3} \pi r^3 \). By applying the Chain Rule, we have

\[
\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}
\]

\[
\Rightarrow \frac{dr}{dt} = \frac{1}{4\pi r^2} \cdot \frac{dV}{dt}
\]

Substitute \( \frac{dV}{dt} = 100 \) into the eq. we have

\[
\frac{dr}{dt} = \frac{1}{4\pi (25^2)} \cdot 100 = \frac{25}{\pi (25)}
\]

When the diameter is 50 cm, the radius is \( \frac{50}{2} = 25 \) cm.

Therefore, \( \frac{dr}{dt} \bigg|_{r=25} = \frac{25}{\pi (25^2)} = \frac{1}{25\pi} \)

The radius of the balloon is increasing at the rate of \( \frac{1}{25\pi} \text{ cm/s} \).
EXAMPLE 2 A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?

Solution:

We are given \( \frac{dx}{dt} = 1 \), we are asked to find \( \frac{dy}{dt} \) when \( x = 6 \).

Since \( x^2 + y^2 = 100 \).

By differentiating each side of the eq. , we have

\[
2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0
\]

\[\Rightarrow \frac{dy}{dt} = -\frac{x}{y} \frac{dx}{dt}\]

Since \( \frac{dx}{dt} = 1 \), \( \frac{dy}{dt} = -\frac{x}{y} \).

When \( x = 6 \), \( y = 8 \). Substitute these values,

we have \( \frac{dy}{dt} \bigg|_{x=6} = -\frac{6}{8} = -\frac{3}{4} \).

The top of the ladder is sliding down the wall at a rate of \( \frac{3}{4} \) ft/s.
EXAMPLE 3 A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 \( m^3/\text{min} \), find the rate at which the water level is rising when the water is 3 m deep.

We are given \( \frac{dV}{dt} = 2 \ m^3/\text{min} \), and we are asked to find

\[
\frac{dh}{dt} \bigg|_{h=3}
\]

\[
V = \frac{1}{3} \pi r^2 h
\]

and \( \frac{h}{4} = \frac{r}{2} \Rightarrow r = \frac{1}{2} h \)

\[
V = \frac{1}{3} \pi \left( \frac{1}{2} h \right)^2 h = \frac{\pi}{12} h^3
\]

Differentiate both sides w.r.t. \( t \), we get

\[
\frac{dV}{dt} = \frac{\pi}{12} \cdot 3 \cdot h^2 \cdot \frac{dh}{dt} = \frac{\pi}{4} h^2 \cdot \frac{dh}{dt}
\]

\[
\Rightarrow \frac{dh}{dt} = \frac{4}{\pi h^2} \cdot \frac{dV}{dt} = \frac{4}{\pi h^2} \cdot 2 = \frac{8}{\pi h^2} \quad (\because \frac{dV}{dt} = 2)
\]

Therefore,

\[
\frac{dh}{dt} \bigg|_{h=3} = \frac{8}{\pi (3)^2} = \frac{8}{9\pi}
\]

The water level is rising at a rate of \( \frac{8}{9\pi} \ m/\text{min} \).
EXAMPLE 4 Car A is traveling west at 50 mi/h and car B is traveling north at 60 mi/h. Both are headed for the intersection of the two roads. At what rate are the cars approaching each other when car A is 0.3 mi and car B is 0.4 mi from the intersection?

1) We are given \( \frac{dx}{dt} = -50 \text{ mi/h} \) and \( \frac{dy}{dt} = -60 \text{ mi/h} \), and we are asked to find \( \frac{dz}{dt} \) when \( x = 0.3 \) and \( y = 0.4 \).

\[ Z^2 = X^2 + Y^2 \]

Differentiate both sides w.r.t. \( t \), we have

\[ 2Z \frac{dz}{dt} = 2X \frac{dx}{dt} + 2Y \frac{dy}{dt} \]

Substitute \( \frac{dx}{dt} = -50 \) and \( \frac{dy}{dt} = -60 \) in, we have

\[ \frac{dz}{dt} = \frac{1}{Z} \left[ X \cdot (-50) + Y \cdot (-60) \right] \]

When \( x = 0.3 \) and \( y = 0.4 \), \( Z = 0.5 \)

\[ \frac{dz}{dt} \bigg|_{x=0.3, \ y=0.4, \ z=0.5} = \frac{1}{0.5} \left[ (0.3)(-50) + (0.4)(-60) \right] = -78 \]

The cars are approaching each other at a rate of 78 mi/h.
EXAMPLE 5 A man walks along a straight path at a speed of 4 ft/s. A searchlight is located on the ground 20 ft from the path and is kept focused on the man. At what rate is the searchlight rotating when the man is 15 ft from the point on the path closest to the searchlight?

Solution:

We are given \( \frac{dx}{dt} = 4 \text{ ft/s} \), and we are asked to find \( \frac{d\theta}{dt} \) when \( x = 15 \).

Since \( \tan \theta = \frac{x}{20} \). By differentiating both sides w.r.t. \( t \), we have

\[
\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{20} \frac{dx}{dt}
\]

Substitute \( \frac{dx}{dt} = 4 \text{ ft/s} \) into the equation, we get

\[
\sec^2 \theta \frac{d\theta}{dt} = \frac{1}{20} \cdot 4
\]

\[
\Rightarrow \quad \frac{d\theta}{dt} = \frac{1}{5} \cdot \frac{1}{\sec \theta} = \frac{1}{5} \cos^2 \theta
\]

When \( x = 15 \), \( \tan \theta = \frac{15}{20} = \frac{3}{4} \). \( \Rightarrow \cos \theta = \frac{4}{5} \).

Therefore, \( \frac{d\theta}{dt} \bigg|_{x=15} = \frac{1}{5} \cdot \left(\frac{4}{5}\right)^2 = \frac{16}{125} \)

The searchlight is rotating at a rate of \( \frac{16}{125} \text{ rad/s} \). 

3.11 Linear Approximations and Differentials

The equation of the tangent line is
\[ y = f(a) + f'(a)(x-a) \]

The approximation \[ f(x) \approx f(a) + f'(a)(x-a) \]
is called the linear approximation of \( f \) at \( a \)
(or tangent line approximation of \( f \) at \( a \)).

The linear function \[ L(x) = f(a) + f'(a)(x-a) \] is called the linearization of \( f \) at \( a \).

The graph of \( L(x) \) is the tangent line.

Exercise 1]
Suppose that after you stuff a turkey its temperature is 50°F and you then put it in a 325°F oven. After an hour the meat thermometer indicates that the temperature of the turkey is 93°F and after two hours it indicates 129°F. Predict the temperature of the turkey after three hours.

Solution]
Let \( T(t) \) represent the temperature of the turkey after \( t \) hours.

Then \( T(0) = 50 \), \( T(1) = 93 \) and \( T(2) = 129 \).
In order to make a linear approximation with $a=2$, we need to estimate $T'(2)$

\[ T'(2) = \lim_{t \to 2} \frac{T(t) - T(2)}{t - 2} \quad \Rightarrow \quad T'(2) \approx \frac{T(1) - T(2)}{1 - 2} = \frac{93 - 129}{-1} = 36 \]

Hence, the linear approximation of $T$ at $t=2$ is

\[ T(t) \approx T(2) + T'(2)(t-2) \]

Therefore \[ T(3) \approx T(2) + T'(2)(3-2) = 129 + 36 \cdot 1 = 165 \, ^\circ F \]

1] Find the linearization of the function $f(x) = \sqrt{x+3}$ at $a=1$ and use it to approximate numbers $\sqrt{3.98}$ and $\sqrt{4.05}$. Are these approximations overestimates or underestimates?

\[ f'(x) = \frac{1}{2 \sqrt{x+3}} \quad \Rightarrow \quad f'(1) = \frac{1}{4} \]

The linearization is \[ L(x) = f(1) + f'(1)(x-1) = 2 + \frac{1}{4}(x-1) = \frac{7}{4} + \frac{x}{4} \]

The corresponding linear approximation is \[ \sqrt{x+3} \approx \frac{7}{4} + \frac{x}{4} \]

\[ \sqrt{3.98} \approx \frac{7}{4} + \frac{0.98}{4} = 1.995 \]

\[ \sqrt{4.05} \approx \frac{7}{4} + \frac{1.05}{4} = 2.0125 \]

These approximations are overestimates because the tangent line lies above the curve.
If $y = f(x)$ (f is differentiable), the differential $dy$ is defined as

$$dy = f'(x) \, dx$$

$dx$ is an independent variable, $dy$ is a dependent variable.

Let $dx = \Delta x$

$\Delta y = f(x + \Delta x) - f(x)$

$$dy = f'(x) \, dx$$

$dy$ represents the amount that the tangent line rises or falls (the change in the linearization)

$\Delta y$ represents the amount that the curve $y = f(x)$ rises or falls (the change in $f(x)$) when $x$ changes by an amount $dx$
Compare the values of $\Delta y$ and $dy$ if $y = f(x) = x^3 + x^2 - 2x + 1$ and $x$ changes from $2$ to $2.05$ and (b) from $2$ to $2.01$.

(a) 
(i) $f(2) = 2^3 + 2^2 - 2 \cdot 2 + 1 = 9$

$f(2.05) = (2.05)^3 + (2.05)^2 - 2(2.05) + 1 = 9.717625$

$\Delta y = f(2.05) - f(2) = 0.717625$

(ii) $dy = f'(x)dx = (3x^2 + 2x - 2)dx$

when $x = 2$ and $dx = 2.05 - 2 = 0.05$

$dy = (3 \cdot 2^2 + 2 \cdot 2 - 2) \cdot (0.05) = 0.7$

(b) 
(i) $f(2.01) = (2.01)^3 + (2.01)^2 - 2(2.01) + 1 = 9.140701$

$\Delta y = f(2.01) - f(2) = 0.140701$

(ii) $dy = f'(x)dx = (3x^2 + 2x - 2)dx$

when $x = 2$ and $dx = 2.01 - 2 = 0.01$

$dy = (3 \cdot 2^2 + 2 \cdot 2 - 2) \cdot (0.01) = 0.14$

The approximation $\Delta y \approx dy$ becomes better as $\Delta x$ becomes smaller.
In the notation of differentials, the linear approximation can be written as
\[ f(a + dx) \approx f(a) + dy \]

In example 2. \( f(x) = \sqrt{x+3} \).
\[ dy = f'(x) dx = \frac{1}{2\sqrt{x+3}} \cdot dx \]
If \( a=1 \), \( dx = \Delta x = 0.05 \), then
\[ dy = \frac{1}{2\sqrt{1+3}} \cdot 0.05 = 0.0125 \]
So \( \sqrt{4.05} = f(1.05) \approx f(1) + dy = 2.0125 \)
as we've gotten in example 2.

5] The radius of a sphere was measured and found to be 21 cm with a possible error measurement of at most 0.05 cm. What is the maximum error in using this value of the radius to compute the volume of the sphere?

\[ V = \frac{4}{3} \pi r^3 \]
If the error in \( r \) is \( dr = \Delta r \), then the corresponding error in \( V \) is \( \Delta V \)
\[ \Delta V \approx dV = 4\pi r^2 dr \]
When \( r = 21 \) and \( dr = 0.05 \),
\[ dV = 4\pi (21)^2 \cdot 0.05 \approx 277 \]
The maximum error in the calculated volume is about 277 cm³.

Relative error:
\[ \frac{\Delta V}{V} \approx \frac{dV}{V} = \frac{4\pi r^2 dr}{\frac{4}{3} \pi r^3} = \frac{3dr}{r} \]
The relative error in the radius is
\[ \frac{\Delta r}{r} = \frac{0.05}{21} \approx 0.0024 = 0.24\% \]
and it produces a relative error in the volume of about
\[ \frac{0.7}{0.007} = 0.7\% \]