1.1 Maximum and Minimum Values.

Definition:

1. (a) A function $f$ has an **absolute maximum** (or **global maximum**) at $x = c$ if $f(c) \geq f(x)$ for all $x$ in $D_f$.

   The number $f(c)$ is called the **maximum value** of $f$ on $D_f$.

   (b) $f$ has an **absolute minimum** (or **global minimum**) at $x = c$ if $f(c) \leq f(x)$ for all $x$ in $D_f$.

   The number $f(c)$ is called the **minimum value** of $f$ on $D_f$.

   The maximum and minimum values of $f$ are called the **extreme values** of $f$.

2. (a) A function $f$ has a **local maximum** (or **relative maximum**) at $x = c$ if $f(c) \geq f(x)$ when $x$ is near $c$ (for all $x$ in some open interval containing $c$).

   The number $f(c)$ is called the **local maximum value** of $f$.

   (b) $f$ has a **local minimum** (or **relative minimum**) at $x = c$ if $f(c) \leq f(x)$ when $x$ is near $c$.

   The number $f(c)$ is called the **local minimum value** of $f$.
The function $f(x)$ has an **absolute maximum** at $x=e$, the absolute maximum value $= f(e)$.

- $f(x)$ has an **absolute minimum** at $x=a$, the absolute minimum value $= f(a)$.
- $f(x)$ has a **local maximum** at $x=c$, the local maximum value $= f(c)$.
- $f(x)$ has a **local maximum** at $x=e$, the local maximum value $= f(e)$.
- $f(x)$ has a **local minimum** at $x=d$, the local minimum value $= f(d)$.
- $f(x)$ has a **local minimum** at $x=b$, the local minimum value $= f(b)$.

The absolute minimum is not a local minimum because it occurs at an endpoint.
1. The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times. It also takes on its (local and absolute) minimum value of -1 infinitely many times.

2. $f(x) = x^2$ has an absolute (and local) minimum value $f(0) = 0$. It has no maximum value.

**The Extreme Value Theorem**

If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$. 

![Graphs showing the Extreme Value Theorem](image)
If $f$ is not continuous on the closed interval $[a, b]$, it may not have extreme values. (as shown below)

$f$ has an absolute min. value $= f(b)$, but no maximum value.

This continuous function $g$ has no extreme values.

**Def**

A critical number of a function $f$ is a number $c \in D_f$ s.t.

- either $f'(c) = 0$
- or $f'(c)$ does not exist.

**Ex.** If $f(x) = x^3$ then $f'(x) = 3x^2$. $f'(x) = 0 \Leftrightarrow 3x^2 = 0 \Rightarrow x = 0$. \therefore $x = 0$ is a critical number of $f$.

**Ex.** $x = 0$ is the critical number of $f(x) = |x|$ since $f'(0)$ does not exist.
Find the critical numbers of \( f(x) = x^{\frac{3}{5}} (4-x) \)

**Solution:**

\[
f'(x) = \frac{3}{5} x^{-\frac{2}{5}} (4-x) + \frac{3}{5} x^{\frac{3}{5}} = \frac{3(4-x) - 5x}{5x^{\frac{2}{5}}} = \frac{12 - 8x}{5x^{\frac{2}{5}}} \]

\[
f'(x) = 0 \iff 12 - 8x = 0 \iff x = \frac{3}{2} \]

\( f'(0) \) D.N.E \iff \( x = 0 \)

Thus, the critical numbers are \( x = \frac{3}{2} \) and \( x = 0 \).

The critical numbers of \( f \) are \( x = c, x = e, x = l \) (\( f'(x) = 0 \)) and \( x = d \) (\( f'(x) \) D.N.E).

Note that the local max. and min. occur at these points.
**Thm**

If $f$ has a local max. or min. at $x = C$, then $C$ is a critical number of $f$.

The theorem asserts that every local max. or min. occur at a critical number.

But, be careful! The converse is false!! That means it may happen that $C$ is a critical number of $f$, but $f$ has no local max. or min. at $x = C$.

**Ex.** \[ f(x) = x^3. \]

$x = 0$ is a critical number of $f$ since $f'(0) = 0$, but there's no local max. or min. at $x = 0$. $f'(0) = 0$ simply means that the curve $y = x^3$ has a horizontal tangent at $x = 0$.

**Ex.** \[ f(x) = x^{\frac{1}{3}} \]

$x = 0$ is a critical number of $f$ since $f'(0)$ D.N.E. but $f$ has no local max. or min. at $x = 0$. Here, "$f'(0)$ D.N.E." simply means that the curve $y = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$.
The Closed Interval Method

To find the absolute maximum and minimum values of a continuous function \( f \) on a closed interval \([a,b]\).

1. Find the values of \( f \) at the critical numbers of \( f \) in \((a,b)\).
2. Find the values of \( f \) at the endpoints of the interval.
3. The largest of the values from Steps 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Find the absolute maximum and minimum values of the function
\[
f(x) = x^3 - 3x^2 + 1 \quad , \quad -\frac{1}{2} \leq x \leq 4
\]

01] Since \( f \) is continuous on the closed interval \([-\frac{1}{2}, 4]\), we can use the Closed Interval Method

1. \( f'(x) = 3x^2 - 6x = 3x(x-2) \), \( f'(x)=0 \iff x=0 \) or \( x=2 \) ← critical numbers

   \( f(0)=1 \), \( f(2)=-3 \)

2. \( f\left(-\frac{1}{2}\right)=\frac{1}{8} \), \( f(4)=17 \)

3. the absolute maximum value = \( f(4)=17 \)
   the absolute minimum value = \( f(2)=-3 \)
Exercise 7: Find the absolute maximum and minimum values of the function

\[ f(x) = x - 2 \sin x \quad , \quad 0 \leq x \leq \pi \]

Sol:  
1. \( f'(x) = 1 - 2 \cos x \quad , \quad f'(x) = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} \) or \( x = \frac{5\pi}{3} \)
   
   \[ f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2 \sin \frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} < 0 \]
   
   \[ f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2 \sin \frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.96 \]

2. \( f(0) = 0 \quad ; \quad f(2\pi) = 2\pi \approx 6.28 \)

3. The absolute maximum = \( f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \)
   
   The absolute minimum = \( f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \)

Exercise 8: Find the absolute maximum and minimum of \( f(x) = 16 - 4x \) on \([-3, 3]\)

Sol:  
1. \( f'(x) = \begin{cases} -4 \quad , \quad \text{when } x < \frac{3}{2} \\ 4 \quad , \quad \text{when } x > \frac{3}{2} \end{cases} \) and \( f'\left(\frac{3}{2}\right) \) D.N.E. \( \therefore \frac{3}{2} \) is a critical number
   
   \[ f\left(\frac{3}{2}\right) = 0 \]

2. \( f(-3) = 18 \quad , \quad f(3) = 6 \)

3. The absolute maximum = \( f(-3) = 18 \)
   
   The absolute minimum = \( f\left(\frac{3}{2}\right) = 0 \)

\[ \text{Graph of } y = 16 - 4x \]
4.2 The Mean Value Theorem

Rolle's Thm

Let $f$ be a function that satisfies the following three hypothesis:
1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.
3. $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$

Proof:
Case I: If $f(x) = k$ (a constant), then $f'(x) = 0 \ \forall x \in (a, b)$ [Fig. (1)]

Case II: If $f(x) > f(a)$ for some $x \in (a, b)$. [Fig. (2) and (3)]
Since $f$ is cont. on a closed interval, by the Extreme Value Theorem, $f$ has a maximum value somewhere in $[a, b]$. Because $f(a) = f(b)$, the max. value must occur at a number $c$ in $(a, b)$.
That is, $f(c)$ is a local max. value. Since $f$ is diff. at $c$ by hypothesis 2, we have $f'(c) = 0$.

Case III: If $f(x) < f(a)$ for some $x \in (a, b)$. [Fig. (4)]
Similarly, $f$ has a min. value in $[a, b]$. Since $f(a) = f(b)$, the min. value must occur at a number $c$ in $(a, b)$. And therefore $f(c)$ is a local min. value.
Again, $f'(c) = 0$ since $f$ is diff. at $c$. 

(1) Let \( s = f(t) \) stand for the position function of a moving object. If the object is in the same place at two different instants \( t = a \) and \( t = b \), then \( f(a) = f(b) \). Rolle's Thm says there is some instant of time \( t = c \) between \( a \) and \( b \) s.t. \( f'(c) = 0 \), that is, the velocity is 0 (i.e. \( v(c) = 0 \)).

(2) Prove that the eq. \( x^3 + x - 1 = 0 \) has exactly one real root.

(a) Let \( f(x) = x^3 + x - 1 \). Then \( f(0) = -1 < 0 \) and \( f(1) = 1 > 0 \).

Since \( f \) is a polynomial, \( f \) is cont. on \([0, 1]\).

By the Intermediate Value Thm, there is a number \( c \in (a, b) \) s.t. \( f(c) = 0 \). Thus, the eq. has a root.

(b) To show that the eq. has exactly one root, we use Rolle's Thm and argue by contradiction.

Suppose that the eq. had two roots \( a \) and \( b \). Then \( f(b) = f(a) = 0 \).

Besides, since \( f \) is a polynomial, it is differentiable on \((a, b)\) and cont. on \([a, b]\).

Thus, by Rolle's Thm, there exists a number \( c \in (a, b) \) s.t. \( f'(c) = 0 \).

But \( f'(x) = 3x^2 + 1 \geq 1 \) for all \( x \). So \( f'(x) \) can never be 0.

This gives a contradiction.

Therefore, the eq. can’t have two real roots. That is, it has exactly one root.
The Mean Value Thm.

Let $f$ be a function that satisfies the following hypotheses:

1. $f$ is continuous on the closed interval $[a, b]$
2. $f$ is differentiable on the open interval $(a, b)$

Then there is a number $c$ in $(a, b)$, s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a)$$
The eq. of the secant line $AB$ is
\[ y - f(a) = \frac{f(b) - f(a)}{b-a} (x-a) \]
or
\[ y = f(a) + \frac{f(b) - f(a)}{b-a} (x-a) \]
Let
\[ h(x) = f(x) - \left[ f(a) + \frac{f(b)-f(a)}{b-a} (x-a) \right] \]
Since $h(x)$ is the sum of $f$ and a first-degree polynomial, both of which are cont. on $[a,b]$ and diff. on $(a,b)$, we know that $h(x)$ is also cont. on $[a,b]$ and diff. on $(a,b)$ and
\[ h'(x) = f'(x) - \frac{f(b)-f(a)}{b-a} \]
Besides, $h(a) = f(a) - \left[ f(a) + \frac{f(b)-f(a)}{b-a} (a-a) \right] = 0$ and $h(b) = f(b) - \left[ f(b) + \frac{f(b)-f(a)}{b-a} (b-a) \right] = 0$

ie. $h(a) = h(b) = 0$

Therefore, by Rolle's Thm., there exists a number $c$ in $(a,b)$ s.t. $h'(c) = 0$

That is,
\[ h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \]

ie. $f'(c) = \frac{f(b)-f(a)}{b-a}$
Consider \( f(x) = x^3 - x \), \( a = 0 \), \( b = 2 \)

Since \( f \) is a polynomial, \( f \) is cont. on \([0,2]\) and diff on \((0,2)\).

Therefore, by Mean Value Theorem, there is a number \( c \in (0,2) \) s.t. \( f(c) - f(0) = f'(c)(2-0) \)

Substitute \( f(2) = 6 \), \( f(0) = 0 \) and \( f'(x) = 3x^2 - 1 \) into the eq.

we get \[ 6 - 0 = (3c^2 - 1)(2-0) \]

\[ 6c^2 = 8 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm \frac{2}{\sqrt{3}} = \pm \frac{2}{\sqrt{3}} \]

But \( c \) must lie in \((0,2)\), so \( c = \frac{2}{\sqrt{3}} \)

The main significance of the Mean Value Thm is that it enable us to obtain information about a function from information about its derivative.

2.5 Suppose that \( f(0) = -3 \) and \( f'(x) \leq 5 \) for all values of \( x \). How large can \( f(2) \) possibly be?

Since \( f(x) \) exists for all \( x \), that is, \( f \) is diff and therefore conti. everywhere.

In particular, we can apply the Mean Value Thm on the interval \([0,2]\).

There exists a number \( c \in (0,2) \) s.t. \( f(2) - f(0) = f'(c)(2-0) \)

\[ f(2) = f(0) + 2f'(c) = -3 + 2f'(c) \leq -3 + 2 \cdot 5 = 7 \]

The largest possible value for \( f(2) \) is 7.
If \( f(x) = 0 \) for all \( x \) in an interval \( (a, b) \), then \( f \) is constant on \( (a, b) \).

**Proof:**

Let \( x_1 \) and \( x_2 \) be any two numbers in \( (a, b) \) with \( x_1 < x_2 \).

Since \( f \) is diff. and therefor cont. on \( (a, b) \), it must be diff. on \( (x_1, x_2) \) and cont. on \([x_1, x_2] \).

By applying the Mean Value Theorem to \( f \) on the interval \([x_1, x_2] \), we know that there is a number \( c \) s.t. \( f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \).

Since \( f'(c) = 0 \), we have \( f(x_2) - f(x_1) = 0 \) i.e. \( f(x_2) = f(x_1) \).

Therefore, \( f \) has the same value at any two numbers in \( (a, b) \).

This means \( f \) is constant on \( (a, b) \).

**Corollary 7**

If \( f'(x) = g'(x) \) for all \( x \) in an interval \( (a, b) \), then \( f-g \) is constant on \( (a, b) \); that is, \( f(x) = g(x) + C \) where \( C \) is a constant.

**Proof:**

Let \( F(x) = f(x) - g(x) \). Then \( F'(x) = f'(x) - g'(x) = 0 \) for all \( x \) in \( (a, b) \).

Thus, by Theorem 5, we conclude that \( F \) is constant, i.e. \( f-g \) is constant.
6] Prove the identity \( \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \)

Let \( f(x) = \tan^{-1} x + \cot^{-1} x \)

Then \( f'(x) = \frac{1}{1+x^2} - \frac{1}{1+x^2} = 0 \) for all \( x \)

Therefore \( f(x) = C \) where \( C \) is a constant.

To determine the value of \( C \), we substitute 1 for \( x \) into the eq.

\[ C = f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \]

Thus, \( \tan^{-1} x + \cot^{-1} x = \frac{\pi}{2} \)
4.3 How Derivatives Affect the Shape of a Graph

Increasing/Decreasing Test (I/D Test)

(a) If $f'(x) > 0$ on an interval, then $f$ is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then $f$ is decreasing on that interval.

f1:

Let $x_1$ and $x_2$ be any two numbers in the interval with $x_1 < x_2$.

Since $f$ is diff. (and therefore cont.) on that interval, we know $f$ is diff. on $(x_1, x_2)$ and cont. on $[x_1, x_2]$. So by the Mean Value Thm, there is a number $c \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0 \quad (\because f'(c) > 0 \text{ and } x_2 - x_1 > 0)$$

i.e. $f(x_2) > f(x_1)$.

Part (b) is proved similarly.

f2: Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing.

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2) = 12x(x - 2)(x + 1); \quad f'(x) = 0 \iff x = 0, 2, -1$$

<table>
<thead>
<tr>
<th>Interval</th>
<th>$f'(x)$</th>
<th>$f$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x &lt; -1$</td>
<td>$-$</td>
<td>$\uparrow$ on $(-\infty, -1)$</td>
</tr>
<tr>
<td>$-1 &lt; x &lt; 0$</td>
<td>$+$</td>
<td>$\uparrow$ on $(-1, 0)$</td>
</tr>
<tr>
<td>$0 &lt; x &lt; 2$</td>
<td>$-$</td>
<td>$\downarrow$ on $(0, 2)$</td>
</tr>
<tr>
<td>$x &gt; 2$</td>
<td>$+$</td>
<td>$\downarrow$ on $(2, \infty)$</td>
</tr>
</tbody>
</table>

So $f$ is increasing ($\uparrow$) on $(-1, 0)$ and $(2, \infty)$ and it is decreasing ($\downarrow$) on $(-\infty, -1)$ and $(0, 2)$.
The First Derivative Test

Suppose that \( x = c \) is a critical number of a continuous function \( f \).

(a) If \( f' \) changes from positive to negative at \( x = c \), then \( f \) has a local max. at \( x = c \).
(b) If \( f' \) changes from negative to positive at \( x = c \), then \( f \) has a local min. at \( x = c \).
(c) If \( f' \) does not change sign at \( x = c \), then \( f \) has no local max. or min. at \( x = c \).

27) Find the local min. and max. values of the function \( f \) in \( \mathbb{R} \).

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

1. Since \( f'(x) = 12x^3 - 12x^2 - 24x \),
   - \( f'(-1) = 0 \) is a local min. value.

2. Since \( f'(x) = 12x^3 - 12x^2 - 24x \),
   - \( f'(0) = 5 \) is a local max. value.

3. Since \( f'(x) = 12x^3 - 12x^2 - 24x \),
   - \( f'(2) = -27 \) is a local min. value.
Find the local max. and min. values of the function
\[ g(x) = x + 2 \sin x, \quad 0 \leq x \leq 2\pi \]

\[ g'(x) = 1 + 2 \cos x. \]
\[ g'(x) = 0 \iff \cos x = -\frac{1}{2} \Rightarrow x = \frac{2\pi}{3} \text{ and } \frac{4\pi}{3} \leftarrow \text{critical numbers}. \]

<table>
<thead>
<tr>
<th>Interval</th>
<th>( g' )</th>
<th>( g )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 &lt; x &lt; \frac{2\pi}{3} )</td>
<td>+</td>
<td>( \uparrow ) on ( (0, \frac{2\pi}{3}) )</td>
</tr>
<tr>
<td>( \frac{2\pi}{3} &lt; x &lt; \frac{4\pi}{3} )</td>
<td>-</td>
<td>( \downarrow ) on ( (\frac{2\pi}{3}, \frac{4\pi}{3}) )</td>
</tr>
<tr>
<td>( \frac{4\pi}{3} &lt; x &lt; 2\pi )</td>
<td>+</td>
<td>( \uparrow ) on ( (\frac{4\pi}{3}, 2\pi) )</td>
</tr>
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</table>

By the First Derivative Test,
the local max. value = \[ g\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2\sin\frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \]
the local min. value = \[ g\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2\sin\frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(\frac{-\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \]
What Does $f''$ Say about $f$?

**Def**

1. If the graph of $f$ lies above all of its tangent lines on an interval $I$, then it is called **concave upward** (CU) on $I$.
2. If the graph of $f$ lies below all of its tangent lines on an interval $I$, then it is called **concave downward** (CD) on $I$.

**Concavity Test**

(a) If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward (CU) on $I$.

(b) If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward (CD) on $I$. 
Def
A point $P$ on the curve $y = f(x)$ is called an **inflection point** if $f$ is continuous and the curve changes from concave upward to concave downward or from concave downward to concave upward at $P$.

Sketch a possible graph of a function $f$ that satisfies the following conditions:
(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$
(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$
(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

By (i), we know that $f$ is $C^1$ on $(-\infty, 1)$ and $2$ on $(1, \infty)$.
By (ii), we know that $f$ is $C^2$ on $(-\infty, -2)$ and $(2, \infty)$ and $f$ is $C^3$ on $(-2, 2)$.
By (iii), we know that $y = -2$ and $y = 0$ are horizontal asymptotes of $y = f(x)$.
The Second Derivative Test

Suppose $f''$ is continuous near $c$

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local min. at $x = c$
(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local max. at $x = c$

6] Discuss the curve $y = x^4 - 4x^3$ w.r.t. concavity, points of inflection, and local max. or min. Use this information to sketch the curve.

[1]: If $f(x) = x^4 - 4x^3$, then $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$ and $f''(x) = 12x^2 - 24x - 12x(x - 2)$. Therefore $f'(x) = 0 \Rightarrow x = 0, x = 3$ (critical numbers)

Since $f''(3) = 36 > 0$, $f(3) = -27$ is a local min.

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0. But, by the First Derivative Test, since $f'(x) < 0$ for $x < 0$ and $0 < x < 3$, $f$ has no local max. or min. at 0.

Set $f''(x) = 0 \Rightarrow x = 0, x = 2$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f'(x)$</th>
<th>$f''(x)$</th>
<th>$f''(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$CU$</td>
</tr>
<tr>
<td>$(0, 2)$</td>
<td>$-$</td>
<td>$+$</td>
<td>$CD$</td>
</tr>
<tr>
<td>$(2, \infty)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$CU$</td>
</tr>
</tbody>
</table>

Therefore, the inflection points are $(0, 0)$ and $(2, -16)$.
Note: (1) The Second Derivative Test is inconclusive when $f''(c) = 0$. It gives no information about the critical number $c$ if $f''(c) = 0$.

So when $f'(c) = 0$ and $f''(c) = 0 \rightarrow$ Use the First Derivative Test.

(2) The Second Derivative Test fails when $f''(c) \text{ D.N.E.} \rightarrow$ Use the First Derivative Test.

7] Sketch the graph of the function $f(x) = x^{\frac{2}{3}} (6-x)^{\frac{1}{3}}$

01]:

$f'(x) = \frac{4-x}{x^{\frac{1}{3}} (6-x)^{\frac{2}{3}}}$

$f''(x) = \frac{-8}{x^{\frac{2}{3}} (6-x)^{\frac{5}{3}}}$

$f'(x) = 0 \iff x = 4$

$f'(x) \text{ D.N.E.} \iff x = 0, x = 6$

$\therefore x = 0, 4, 6$ are critical numbers.

| $(-\infty, 0)$ | $-\downarrow$ | $\downarrow$ |
| $(0, 4)$ | $+$ | $\uparrow$ |
| $(4, 6)$ | $-\downarrow$ | $\downarrow$ |
| $(6, \infty)$ | $-$ | $\downarrow$ |

$f''(x)$ D.N.E. $\iff x = 0, x = 6$

| $(-\infty, 0)$ | $-$ | CD |
| $(0, 6)$ | $-$ | CD |
| $(6, \infty)$ | $+$ | CU |

$\therefore$ the point of inflection is $(6, 0)$

$f(0) = 0$ is a local min.

$f(4) = 2^{\frac{5}{3}}$ is a local max.

$f(x)$ has no local max. or min at $x = 6$.
Use the first and second derivative of \( f(x) = e^{\frac{1}{x}} \), together with asymptotes, to sketch its graph.

1. \[ f'(x) = -\frac{e^{\frac{1}{x}}}{x^2} \]

   \[
   \begin{array}{c|cc}
   & f' & f \\
   \hline
   (-\infty, 0) & - & \downarrow \\
   (0, \infty) & - & \downarrow \\
   \end{array}
   \]

   \[ f(x) \text{ D.N.E.} \Leftrightarrow x = 0 \]

   \( \therefore f \) has no local max. or min.

2. \[ f''(x) = \frac{e^{\frac{1}{x}}(2x+1)}{x^4} \]

   \[ f''(x) = 0 \Leftrightarrow x = -\frac{1}{2} \]

   \[ f''(x) \text{ D.N.E.} \Leftrightarrow x = 0 \]

   \[
   \begin{array}{c|cc}
   & f'' & f \\
   \hline
   (-\infty, -\frac{1}{2}) & - & \text{CD} \\
   (-\frac{1}{2}, 0) & + & \text{CU} \\
   (0, \infty) & + & \text{CU} \\
   \end{array}
   \]

   \( \therefore \) the inflection point is \((-\frac{1}{2}, e^{2})\)

3. \[ \lim_{x \to 0^+} e^{\frac{1}{x}} = \infty \quad \therefore x = 0 \text{ is a vertical asymptote} \]

   \( \lim_{x \to 0^-} e^{\frac{1}{x}} = 0 \)

   \[ \lim_{x \to \pm\infty} e^{\frac{1}{x}} = e^0 = 1 \quad \therefore y = 1 \text{ is a horizontal asymptote.} \]
3.4.4 Indeterminate Forms and L'Hôpital's Rule.

If $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$, then the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called an indeterminate form of type $0 \over 0$.

If $\lim_{x \to a} f(x) = \infty$ (or $-\infty$) and $\lim_{x \to a} g(x) = \infty$ (or $-\infty$), then the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ is called an indeterminate form of type $\frac{\infty}{\infty}$.

L'Hôpital's Rule

Suppose $f$ and $g$ are differentiable and $g(x) \neq 0$ near $a$ (except possibly at $a$).

Suppose that $\lim_{x \to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $0 \over 0$ or $\frac{\infty}{\infty}$.

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$ if the limit $\lim_{x \to a} \frac{f(x)}{g(x)}$ exists (or is $\infty$ or $-\infty$).

Note 1: It is especially important to verify the conditions regarding the limits of $f$ and $g$ before using L'Hôpital's Rule.

Note 2: L'Hôpital's Rule also valid for one-sided limit and for limits at infinity or negative infinity; that is, "$x \to a$" can be replace by $x \to a^+$, $x \to a^-$, $x \to \infty$ or $x \to -\infty$. 


[x1] Find \( \lim_{x \to 1} \frac{\ln x}{x - 1} \)

So 1] The limit is an indeterminate form of type \( \frac{0}{0} \), we can apply L'Hôpital's Rule:

\[
\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1}{1} = 1
\]

[x2] Calculate \( \lim_{x \to 0} \frac{e^x}{x^2} \)

So 1] The limit is an indeterminate form of type \( \frac{\infty}{\infty} \), we can apply L'Hôpital's Rule:

\[
\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(2x)} = \lim_{x \to \infty} \frac{e^x}{2} = \infty
\]

\[\left( \lim_{x \to \infty} \frac{e^x}{x^2} \text{ is still an indeterminate form of type } \frac{\infty}{\infty} \right)\]

So, use the L'Hôpital's Rule again.

[x3] Calculate \( \lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} \)

So 1] Apply L'Hôpital's Rule to it because it's an indeterminate form of type \( \frac{\infty}{\infty} \).

\[
\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^{\frac{1}{3}})} = \lim_{x \to \infty} \frac{1}{\frac{1}{3}x^{\frac{2}{3}}} = \lim_{x \to \infty} \frac{3}{\sqrt[3]{x}} = 0
\]
x4] Find \( \lim_{x \to 0} \frac{\tan x - x}{x^3} \)

01] It's of the type \( \frac{0}{0} \), so we can apply L'Hôpital's Rule:

\[
\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2 \sec x \cdot \sec x \cdot \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{x} \\
(\text{type } \frac{0}{0}, \text{ use L'Hôpital's Rule again})
\]

\[
= \frac{1}{3} \lim_{x \to 0} \sec^2 x \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \frac{1}{3} \cdot 1 \cdot 1 = \frac{1}{3}
\]

<4] Find \( \lim_{x \to \infty} \frac{x}{x + \sin x} \)

01] \[
\lim_{x \to \infty} \frac{x}{x + \sin x} \neq \lim_{x \to \infty} \frac{1}{1 + \cos x} \quad \text{the limit does not exist.}
\]

\[
\lim_{x \to \infty} \frac{x}{x + \sin x} = \lim_{x \to \infty} \frac{x}{x + \sin x} = \lim_{x \to \infty} \frac{1}{1 + \frac{\sin x}{x}} = 1
\]
Indeterminate Products

If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = \infty \) (or \(-\infty\)), then the limit \( \lim_{x \to a} f(x)g(x) \) is called an indeterminate form of type \( 0 \cdot \infty \).

\[
\lim_{x \to a} f(x)g(x) = \begin{cases} 
\lim_{x \to a} \frac{f(x)}{g(x)} & \text{indeterminate form of type } \frac{0}{\infty} \\
\lim_{x \to a} \frac{g(x)}{f(x)} & \text{indeterminate form of type } \frac{\infty}{0}
\end{cases}
\]

67 Evaluate \( \lim_{x \to 0^+} x \ln x \)

Solution: Since \( \lim_{x \to 0^+} x = 0 \) and \( \lim_{x \to 0^+} \ln x = -\infty \), the limit is an indeterminate form of type \( 0 \cdot \infty \).

Using L'Hôpital's Rule, (but converting the limit into the form of type \( \frac{0}{\infty} \) or \( \frac{\infty}{0} \) first), we have

\[
\lim_{x \to 0^+} x \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{d}{dx} \ln x}{\frac{d}{dx} \left(\frac{1}{x}\right)} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0.
\]
Indeterminate Differences

If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = \infty \), then the limit \( \lim_{x \to a} [f(x) - g(x)] \) is called an indeterminate form of type \( \infty - \infty \).

In this case, we try to convert the difference into a quotient so that we have an indeterminate form of type \( \frac{\infty}{\infty} \) or \( \frac{0}{0} \).

\[ x^7 \] Compute \( \lim_{x \to \frac{\pi}{2}} \left( \sec x - \tan x \right) \)

Solution: This is an indeterminate form of \( \infty - \infty \). We'll try to convert the difference into a quotient.

\[
\lim_{x \to \frac{\pi}{2}} \left( \sec x - \tan x \right) = \lim_{x \to \frac{\pi}{2}} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \to \frac{\pi}{2}} \frac{1 - \sin x}{\cos x} \quad \text{(indeterminate form of type \( \frac{0}{0} \))}
\]

\[ = \lim_{x \to \frac{\pi}{2}} \frac{-\cos x}{-\sin x} \]

\[ = 0 \]
Indeterminate Powers

If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \), then the limit \( \lim_{x \to a} [f(x)]^{g(x)} \) is called an indeterminate form of type \( 0^0 \).

If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = 0 \), then the limit \( \lim_{x \to a} [f(x)]^{g(x)} \) is called an indeterminate form of type \( \infty^0 \).

If \( \lim_{x \to a} f(x) = 1 \) and \( \lim_{x \to a} g(x) = \pm \infty \), then the limit \( \lim_{x \to a} [f(x)]^{g(x)} \) is called an indeterminate form of type \( 1^\infty \).

In these cases, we'll write the function \( [f(x)]^{g(x)} \) as an exponential:
\[
[f(x)]^{g(x)} = e^{\ln[f(x)]^{g(x)}} = e^{g(x) \ln f(x)}
\]

Ex 9] Find \( \lim_{x \to b^+} x^x \)

Sol]: It's an indeterminate form of type \( 0^0 \).
\[
\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\ln x^x} = \lim_{x \to 0^+} e^{x \ln x} = e^{\lim_{x \to 0^+} x \ln x} = e^0 = 1
\]
\( (\because e^x \) is conti. and \( \lim_{x \to 0} x \ln x = 0 \) exists by Ex 6)
Calculate \( \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} \)

**Solution:**

This is an indeterminate form of type \(1^\infty\).

\[
\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} e^{\ln (1 + \sin 4x) \cot x} = \lim_{x \to 0^+} e^{\cot x \cdot \ln (1 + \sin 4x)}
\]

Since \( \lim_{x \to 0^+} \cot x \cdot \ln (1 + \sin 4x) = \lim_{x \to 0^+} \frac{\ln (1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\cos 4x \cdot 4}{1 + \sin 4x} \cdot \frac{4}{\sec^2 x} = 4 \),

we have that \( \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = e^{\lim_{x \to 0^+} \cot x \cdot \ln (1 + \sin 4x)} = e^4 \) (\( e^x \) is a continuous function)
4.5 Summary of Curve Sketching

Guidelines for Sketching a Curve:

1. Domain
2. Intercepts
3. Symmetry:
   (i) \( f(-x) = f(x) \) \( \iff \) \( f \) is an even function
      \( \iff \) the graph of \( f \) is symmetric about the y-axis.
   (ii) \( f(-x) = -f(x) \) \( \iff \) \( f \) is an odd function
      \( \iff \) the graph of \( f \) is symmetric about the origin.
   (iii) \( f(x+p) = f(x) \) for all \( x \in \text{D}_f \), where \( p \) is a positive integer.
      \( \iff \) \( f \) is a periodic function.

D. Asymptotes:
   Find vertical asymptotes or horizontal asymptotes or slant asymptotes

   If \( \lim_{x \to \infty} [f(x) - (ax+b)] = 0 \), then the line \( y = ax + b \) is called a
   slant asymptote.

   [EX] If \( f(x) = x + \frac{1}{x} \), then \( \lim_{x \to \infty} [f(x) - x] = \lim_{x \to \infty} \frac{1}{x} = 0 \).
       Therefore \( y = x \) is a slant asymptote of \( y = f(x) \).
E. Intervals of Increase or Decrease

F. Local Maximum and Minimum Values

G. Concavity and Points of Inflection

H. Sketch the Curve.

\[ y = \frac{2x^2}{x^2 - 1} \]

1. Sketch the curve \( y = \frac{2x^2}{x^2 - 1} \)

2. Let \( f(x) = \frac{2x^2}{x^2 - 1} \)

A. \( D_f = \{ x \mid x \neq \pm 1 \} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \)

B. The \( x \)-intercept = 0 and the \( y \)-intercept = 0

C. Symmetry.
   Since \( f(-x) = f(x) \), \( f \) is even. The curve is symmetric about the \( y \)-axis.

D. Asymptotes.
   \[ \lim_{x \to 1^+} f(x) = \infty, \quad \lim_{x \to 1^-} f(x) = -\infty, \quad \lim_{x \to 1^+} f(x) = -\infty, \quad \lim_{x \to 1^-} f(x) = \infty \]
   \[ x = 1 \text{ and } x = -1 \text{ are vertical asymptotes of the curve } y = \frac{2x^2}{x^2 - 1} \]
   \[ \lim_{x \to \pm\infty} f(x) = 2 \quad \therefore y = 2 \text{ is the horizontal asymptote of the curve} \]
E. Intervals of Increase or Decrease.

\[
\frac{f'(x)}{f(x)} = \frac{4x(x^2-1) - 2x^2 - 2x}{(x^2-1)^2} = \frac{-4x}{(x^2-1)^2}, \quad f'(x) = 0 \Leftrightarrow x = 0 \text{ (critical number)}
\]

| \(x < 0, x \neq -1\) | + | \(\uparrow\) on \((-\infty, -1) \cup (1, 0)\) |
| \(x > 0, x \neq 1\) | - | \(\downarrow\) on \((0, 1) \cup (1, \infty)\) |

F. The local maximum value = \(f(0) = 0\)

G. Concavity and Points of Inflection.

\[
f''(x) = \frac{-4x(x^2-1)^3 - (-4x) \cdot 2(x^2-1) \cdot 2x}{(x^2-1)^4} = \frac{12x^2 + 4}{(x^2-1)^3}
\]

\(f''(x) < 0 \Leftrightarrow x^2 - 1 > 0 \Leftrightarrow x = \pm 1\)

| \(-\infty, -1\) | + | \(CU\) |
| \((-1, 1)\) | - | \(CD\) |
| \((1, \infty)\) | + | \(CU\) |

: There is no inflection point because \(x = \pm 1\) are not in the domain of \(f\).
Sketch the graph of \( f(x) = 5(x-1)^{\frac{3}{5}} - 2(x-1)^{\frac{5}{3}} \)

**A.** \( D_f = \mathbb{R} \)

**B.** Intercepts: \( y = 5(x-1)^{\frac{3}{5}} - 2(x-1)^{\frac{5}{3}} \) when \( x = 0 \), \( y = 7 \)
when \( y = 0 \), \( (x-1)^{\frac{3}{5}} [5-2(x-1)] = 0 \) \( \Rightarrow x = 1 \) or \( \frac{7}{2} \)

**C.** Symmetry: None

**D.** Asymptote: None

**E.** Intervals of Increase or Decrease.
\[
f'(x) = \frac{10(2-x)}{3(x-1)^{\frac{2}{3}}} \]

\( f'(x) = 0 \Leftrightarrow x = 2 \)

\( f'(x) \) D.N.E. \( \Leftrightarrow x = 1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f' )</th>
<th>( f )</th>
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<tbody>
<tr>
<td>( (-\infty, 1) )</td>
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<td>↘</td>
</tr>
<tr>
<td>( (1, 2) )</td>
<td>+</td>
<td>↗</td>
</tr>
<tr>
<td>( (2, \infty) )</td>
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</table>

**F.** The local max. value = \( f(2) = 3 \)
The local min. value = \( f(1) = 0 \)

**G.** Concavity and points of inflection
\[
f''(x) = \frac{10(1-2x)}{9(x-1)^{\frac{5}{3}}} \]

\( f''(x) = 0 \Leftrightarrow x = \frac{1}{2} \)
\( f''(x) \) D.N.E. \( \Leftrightarrow x = 1 \)

\[ \begin{array}{c|c|c}
  x & f'' & f \\
  \hline
  (-\infty, \frac{1}{2}) & + & CU \\
  (\frac{1}{2}, 1) & - & CD \\
  (1, \infty) & - & CD \\
\end{array} \]

\( \text{the inflection point is } (\frac{1}{2}, 3^{\frac{3}{2}}) \)

H.

\[ y = 5(x-1)^{\frac{3}{5}} - 2(x-1)^{\frac{5}{3}} \]
3. Sketch the graph of \( f(x) = xe^x \)

**Solution:**

A. \( D_f = \mathbb{R} \)

B. The x-intercept and y-intercept are both 0

C. Symmetry: None

D. Asymptotes:

- \( \lim_{x \to \infty} xe^x = \infty \)
- \( \lim_{x \to -\infty} xe^x = 0 \)

\( y = 0 \) is the horizontal asymptote

E. Intervals of Increase or Decrease

\( f(x) = xe^x + e^x = (x+1)e^x \)

\( f'(x) = 0 \iff x = -1 \)

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<thead>
<tr>
<th>Interval</th>
<th>( f' )</th>
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</tr>
<tr>
<td>((-1, \infty))</td>
<td>+</td>
<td>↑</td>
</tr>
</tbody>
</table>

F. The local min. value is \( f(-1) = -\frac{1}{e} \)

There's no local max.

G. \( f''(x) = (x+1)e^x + e^x = (x+2)e^x \)

\( f''(x) = 0 \iff x = -2 \)
Sketch the graph of \( f(x) = 2 \cos x + \sin 2x \)

\[ f(x) = 2 \cos x + \sin 2x \]

**Solution:**

A. \( D_f = \mathbb{R} \)

B. The y-intercept is \( f(0) = 2 \)

\[ 2 \cos x + \sin 2x = 0 \]

\[ 2 \cos x (1 + \sin x) = 0 \]

\[ \cos x = 0 \quad \text{or} \quad \sin x = -1 \]

\[ x = \frac{\pi}{2} \quad \text{or} \quad x = \frac{3\pi}{2} \quad (\text{in } [0, 2\pi]) \]

C. \( f \) is neither odd nor even, but \( f(x + 2\pi) = f(x) \) for all \( x \).

Therefore \( f \) is a periodic function with period \( 2\pi \).

We may consider only \( 0 \leq x \leq 2\pi \).

D. Asymptote: None

E. \( f'(x) = -2 \sin x + 2 \cos 2x = -2 \sin x + 2(1 - 2 \sin^2 x) \)

\[ = -2(2 \sin x + \sin x - 1) = -2(\sin x + 1)(2 \sin x - 1) \]

\[ f'(x) = 0 \quad \iff \quad \sin x = -1 \quad \text{or} \quad \sin x = \frac{1}{2} \]

\[ \iff \quad \text{in } [0, 2\pi], \quad x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \]

F. \( f''(x) = -2 \cos x (1 + 4 \sin x) \)

\[ f''(x) = 0 \iff \cos x = 0 \quad \text{or} \quad \sin x = -\frac{1}{2} \]

\[ \Rightarrow \quad x = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha_1, \alpha_2 \]

where \( \alpha_1 = \pi + \sin^{-1}(\frac{1}{2}) \)

\[ \alpha_2 = 2\pi - \sin^{-1}(\frac{1}{2}) \]

\[ \begin{array}{c|c|c}
\alpha_1, \frac{3\pi}{2} & - & CD \\
\frac{\pi}{2}, \alpha_2 & + & CU \\
(\alpha_1, \frac{3\pi}{2}) & - & CD \\
(\frac{3\pi}{2}, \alpha_2) & + & CU \\
(\alpha_1, \frac{3\pi}{2}) & - & CD \\
\end{array} \]

The inflection points are \((\frac{\pi}{2}, 0), (\alpha_1, f(\alpha_1)), (\alpha_2, f(\alpha_2))\)

\[ y = \sin x \]

\[ y = x \]

H. We draw the curve on \([0, 2\pi]\) first, then extend the curve by translation.
Sketch the graph of \( y = \ln(4-x^2) \)

Let \( f(x) = \ln(4-x^2) \)

A. \( D_f = \{ x \mid 4-x^2 > 0 \} = \{ x \mid -2 < x < 2 \} = (-2, 2) \)

B. The y-intercept is \( f(0) = \ln 4 \).
   * The x-intercept:
     \[ \ln(4-x^2) = 0 \Rightarrow 4-x^2 = 1 \Rightarrow x = \pm \sqrt{3} \]

C. Symmetry:
   \( \therefore f(-x) = f(x) \) \( \therefore f \) is an even function
   * The curve is symmetric about the y-axis.

D. Asymptote:
   \[ \lim_{x \to 2^-} \ln(4-x^2) = -\infty \quad \lim_{x \to 2^+} \ln(4-x^2) = -\infty \]
   \( \therefore x = 2 \) and \( x = -2 \) are vertical asymptotes

E. Intervals of Increase or Decrease.
   \[ f'(x) = \frac{-2x}{4-x^2} \]
   \[ f''(x) = 0 \iff x = 0 \]

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<thead>
<tr>
<th>( x )</th>
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<tr>
<td>(-2, 0)</td>
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<tr>
<td>(0, 2)</td>
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F. The local Max. value = \( f(0) = \ln 4 \)

G. Concavity and points of inflection.
   \[ f''(x) = \frac{-2(4-x^2)}{(4-x^2)^2} \cdot (-\infty) \cdot (-\infty) \]
   \[ = \frac{-8+2x^2}{(4-x^2)^2} \]
   * Since \( f''(x) < 0 \) for all \( x \) in \((-2, 2)\)
   * The curve is CD on \((-2, 2)\) and there is no point of inflection.

H. [Graph of \( y = \ln(4-x^2) \)]
Sketch the graph of \( f(x) = \frac{x^3}{x^2 + 1} \)

1. \( D_f = \mathbb{R} \)
2. The \( x \)-intercept and \( y \)-intercept are both 0.
   Since \( f(-x) = -f(x) \), \( f \) is odd and its graph is symmetric about the origin.
4. Asymptotes. 
   \( \therefore f(x) = x - \frac{x}{x^2 + 1} \)
   \( \therefore \lim_{x \to \pm \infty} (f(x) - x) = \lim_{x \to \pm \infty} \left( -\frac{x}{x^2 + 1} \right) = 0 \)
   \( \therefore y = x \) is a slant asymptote
5. Intervals of Increase or Decrease 
   \( f'(x) = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2} \)
   Since \( f'(x) > 0 \) for all \( x \in \mathbb{R}, x \neq 0 \), \( f \) is increasing on \( \mathbb{R} \).
6. There's no local max. or min.

\[ \frac{2x(3-x^2)}{(x^2+1)^3} \]
\[ f(x) = 0 \iff x = 0, x = \pm \sqrt{3} \]
\[
\begin{array}{c|c|c}
\hline
x & f'' & f \\
\hline
(-\infty, -\sqrt{3}) & + & CU \\
(-\sqrt{3}, 0) & - & CD \\
(0, \sqrt{3}) & + & CU \\
(\sqrt{3}, \infty) & - & CD \\
\hline
\end{array}
\]
\( \therefore \) the inflection points are \( (-\sqrt{3}, -\frac{3\sqrt{3}}{4}) \) \( (0, 0) \) \( (\sqrt{3}, \frac{3\sqrt{3}}{4}) \)
**Example 1** A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

**Solution:**

Let $x$ and $y$ be the depth and width of the rectangle (in feet). Then $A = xy$ and $2x + y = 2400 \Rightarrow y = 2400 - 2x$.

Thus, $A = x(2400 - 2x) = 2400x - 2x^2$, $0 \leq x \leq 1200$.

We want to maximize $A$ which is cont. on the closed interval $[0, 1200]$.

$A'(x) = 2400 - 4x$.

$A'(x) = 0 \Leftrightarrow x = 600$ (critical number).

$\Rightarrow A(600) = 7200000$.

$\therefore A(0) = 0$, $A(1200) = 0$.

$\therefore A(600) = 7200000$ is the absolute max. value of the area.

(OR. $A' > 0$ when $x < 600$ and $A' < 0$ when $x > 600$.

$\therefore A$ is on $(0, 600)$ and $\nabla$ on $(600, 1200)$.

Therefore $A(600)$ is a local max. value and also an absolute max. value.

Thus, the rectangular field should be 600 ft deep and 1200 ft wide. $\times$
EXAMPLE 2 A cylindrical can is to be made to hold 1 L of oil. Find the dimensions that will minimize the cost of the metal to manufacture the can.

Suppose the can has radius $r$ and height $h$ (in centimeter).

In order to minimize the cost of the metal, we minimize the total surface area of the can.

$$A = 2\pi r^2 + 2\pi rh$$

The volume is given to be 1 L ($=1000$ cm$^3$). Thus, \( \pi r^2 h = 1000 \)

\[ h = \frac{1000}{\pi r^2} \]

Substitution of this into the expression for $A$ gives

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}$$

To minimize $A$, we have to find the critical number first:

$$A' = 4\pi r - \frac{2000}{r^2} = \frac{4\pi r^3 - 2000}{r^2}$$

So $A' = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$

Since $A'(r) < 0$ when $r < \sqrt[3]{\frac{500}{\pi}}$ and $A'(r) > 0$ when $r > \sqrt[3]{\frac{500}{\pi}}$, we know that $A(r)$ is on $(0, \sqrt[3]{\frac{500}{\pi}})$ and $A$ is on $(\sqrt[3]{\frac{500}{\pi}}, \infty)$.

Therefore, $A$ has an absolute min. at $r = \sqrt[3]{\frac{500}{\pi}}$.

The value of $h$ corresponding to $r = \sqrt[3]{\frac{500}{\pi}}$ is

$$h = \frac{1000}{\pi \left(\sqrt[3]{\frac{500}{\pi}}\right)^2} = 2 \sqrt[3]{\frac{500}{\pi}} = 2r$$

Thus, the radius should be $\sqrt[3]{\frac{500}{\pi}}$ cm and the height should be equal to the diameter.
EXAMPLE 3 Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.

The distance between the point $(1,4)$ and the point $(x,y)$ is

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

Since $(x,y)$ lies on the parabola, we have $y^2 = 2x \Rightarrow x = \frac{y^2}{2}$,

$$d = \sqrt{\left(\frac{y^2}{2} - 1\right)^2 + (y-4)^2} = \sqrt{\frac{1}{4}y^4 - 8y + 17}$$

Instead of minimizing $d$, we minimize $d^2 = \frac{1}{4}y^4 - 8y + 17$

Let $f(y) = \frac{1}{4}y^4 - 8y + 17$

$f'(y) = y^3 - 8$, so $f'(y) = 0 \Leftrightarrow y = 2$.

Observe that $f'<0$ when $y<2$ and $f'>0$ when $y>2$, that is, $f$ is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

Therefore, $f$ has an absolute min. at $y = 2$.

The distance $d$ also has an absolute min. at $y = 2$

When $y = 2$, $x = \frac{y^2}{2} = 2$

Thus, the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. 

*
EXAMPLE 4 A man launches his boat from point A on a bank of a straight river, 3 km wide, and wants to reach point B, 8 km downstream on the opposite bank, as quickly as possible (see Figure 7). He could row his boat directly across the river to point C and then run to B, or he could row directly to B, or he could row to some point D between C and B and then run to B. If he can row 6 km/h and run 8 km/h, where should he land to reach B as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

Let \( x \) be the distance from C to D, then the running distance is \( |DB| = 8-x \), and the rowing distance is \( |AD| = \sqrt{x^2+9} \).

So the total time \( T(x) = \frac{x}{6\sqrt{x^2+9}} + \frac{8-x}{8} \)

\( 0 \leq x \leq 8 \).

So \( T(x) = 0 \Leftrightarrow \frac{x}{6\sqrt{x^2+9}} = \frac{1}{8} \Leftrightarrow 4x = 3\sqrt{x^2+9} \Leftrightarrow 7x^2 = 81 \Leftrightarrow x = \frac{9}{\sqrt{7}} \) in \([0,8] \)

To find the point where the absolute min. occur at, we compare the value of \( T \) at the critical number and the end points 0 and 8.

\( T(0) = 1.5 \), \( T(\frac{9}{\sqrt{7}}) = 1 + \frac{\sqrt{7}}{8} \approx 1.33 \), \( T(8) = \frac{\sqrt{73}}{6} \approx 1.42 \).

Therefore, the absolute min. of \( T \) on the closed interval \([0,8] \) occur at \( x = \frac{9}{\sqrt{7}} \). Thus, the man should land the boat at a point \( \frac{9}{\sqrt{7}} \) km downstream from his starting point.
EXAMPLE 5 Find the area of the largest rectangle that can be inscribed in a semicircle of radius \(r\).

[Sol 1]:

Let \((x,y)\) be the vertex that lies in the first quadrant.
Then the rectangle has sides of lengths \(2x\) and \(y\).
So the area \(A = 2xy\).
\[\Rightarrow A = 2x\sqrt{r^2-x^2} \quad 0 \leq x \leq r\]
\[A' = 2\sqrt{r^2-x^2} - \frac{2x^2}{\sqrt{r^2-x^2}} = \frac{2(r^2-2x^2)}{r^2-x^2}, \text{ so } A'(r_0) = 0 \iff x = \frac{r}{\sqrt{2}}\]
Since \(A(0) = 0\), \(A(r) = 0\) and \(A(\frac{r}{\sqrt{2}}) = \frac{r^2}{2}\), we conclude that \(A(\frac{r}{\sqrt{2}}) = \frac{r^2}{2}\) is the absolute max. of \(A\).
The area of the largest inscribed rectangle is \(r^2\)*

[Sol 2]:

Let \(\theta\) be the angle shown in the figure on the left.
Then the area of the rectangle is
\[A(\theta) = 2r \cos \theta \cdot r \sin \theta = r^2 \sin 2\theta\]
We know that \(\sin \theta > 0\) has a max. value of \(1\) and it occurs when \(2\theta = \frac{\pi}{2}\).
Thus, \(A(\theta)\) has a max. value of \(r^2\) and it occurs when \(\theta = \frac{\pi}{4}\)*
4.9 Newton's Method

To approximate a solution to the eq. $f(x) = 0$, choose an initial approximation $x_1$, and calculate $x_2, x_3, x_4, \ldots$ using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 1, 2, 3, \ldots$$

If the numbers $x_1, x_2, x_3, \ldots$ converge, they converge to a solution of $f(x) = 0$.
Note that $x_{n+1}$ might be a worse approximation than $x_n$ (such as $x_3$ in Fig. 1) when $f'(x_n)$ is close to 0. Then Newton’s Method fails and a better initial approximation $x_1$ should be chosen. (So does when the case in Fig. 2 happens.) Newton’s Method also fails when $f(x_n) = 0$ for some $n$.

In this case, there is no $x_3$ produced.
[EX] Use Newton's Method to find the root to the eq. \( x^3 + 3x + 1 = 0 \) to seven decimal places.

[Sol]

Let \( f(x) = x^3 + 3x + 1 \), then \( f'(x) = 3x^2 + 3 \) and

\[
X_{n+1} = X_n - \frac{f(x_n)}{f'(x_n)} = X_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}
\]

The graph of \( f \) suggests that choose \( x_1 = -0.3 \), then

\[
x_2 = x_1 - \frac{x_1^3 + 3x_1 + 1}{3x_1^2 + 3} \approx -0.3223241
\]

\[
x_3 = x_2 - \frac{x_2^3 + 3x_2 + 1}{3x_2^2 + 3} \approx -0.3221853
\]

\[
x_4 = x_3 - \frac{x_3^3 + 3x_3 + 1}{3x_3^2 + 3} \approx -0.3221853
\]

Since \( x_3 \) and \( x_4 \) agree to seven decimal places, we conclude that the root to \( x^3 + 3x + 1 = 0 \) is about \(-0.3221853\).

Usually you don’t have the graph of \( f \) ready to help you decide the value of the initial approximation. In this case, you can make use of the Intermediate Value Theorem: Since \( f(-1) \cdot f(0) < 0 \), there is a root in the interval \((-1, 0)\). Thus, you can choose \( x_1 = -0.5 \) to be the initial approximation. It’s also a good start.
x2] Use Newton's method to find $\sqrt{2}$ correct to eight decimal places.

Solution:

$\sqrt{2}$ is the root of the eq. $x^2 - 2 = 0$

Let $f(x) = x^2 - 2$. then $f'(x) = 2x$. and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n}$$

If we choose $x_1 = 1$ as the initial approximation, then

$$x_2 \approx 1.16666667$$
$$x_3 \approx 1.12644368$$
$$x_4 \approx 1.11404907$$
$$x_5 \approx 1.112246205$$
$$x_6 \approx 1.112246205$$

Since $x_5$ and $x_6$ agree to eight decimal places, we conclude that $\sqrt{2} \approx 1.112246205$

x3] Find, correct to six decimal places, the root of the eq. $\cos x = x$

Solution:

Let $f(x) = \cos x - x$. then $f'(x) = -\sin x - 1$ and

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

If we choose $x_1 = 1$, then

$$x_2 \approx 0.750363$$
$$x_3 \approx 0.739112$$
$$x_4 \approx 0.739085$$
$$x_5 \approx 0.739085$$

Since $x_4$ and $x_5$ agree to six decimal places, we conclude that the root to this eq. is about 0.739085.
4.10 Antiderivatives

**Def**
A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F'(x) = f(x)$ for all $x$ in $I$.

![Diagram showing differentiation leading to the antiderivative of $f$ and antidifferentiation leading to the derivative of $F$.]

**Ex.** $F(x) = \frac{1}{3}x^3$ and $G(x) = \frac{1}{3}x^3 + 5$ are both antiderivatives of $f(x) = x^2$.

**Thm 1**
If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is $F(x) + C$, where $C$ is an arbitrary constant.

**Ex.** The most general antiderivative of $f(x) = x^2$ is $\frac{1}{3}x^3 + C$. 
Find the most general antiderivative of each of the following functions.

(a) \( f(x) = \sin x \)  
(b) \( f(x) = \frac{1}{x} \)  
(c) \( f(x) = x^n, \ n \neq -1 \)

**Solution:**

(a) \[ \frac{d}{dx} (-\cos x) = \sin x \] \[ \therefore \text{the most general antiderivative is } -\cos x + C \]

(b) \[ \frac{d}{dx} (\ln x) = \frac{1}{x} \] \[ \text{on } (0, \infty) \]

So on the interval \((0, \infty)\), the most general antiderivative of \( f \) is \( \ln x + C \)

Also \[ \frac{d}{dx} (\ln |x|) = \frac{1}{x} \] \[ \text{for all } x \neq 0. \]

\[ \therefore \text{on } (-\infty, 0) \text{ and } (0, \infty), \text{ the most general antiderivative of } f = \frac{1}{x} \text{ is } \ln |x| + C \]

Thus, the general antiderivative of \( f \) is

\[ F(x) = \begin{cases} 
\ln x + C & \text{if } x > 0 \\
\ln(-x) + C & \text{if } x < 0.
\end{cases} \]

(c) \[ \therefore \text{when } n \neq -1, \frac{d}{dx} \left( \frac{1}{n+1} x^{n+1} \right) = x^n \]

\[ \therefore \text{the most general antiderivative of } f(x) = x^n \text{ is} \]

\[ F(x) = \frac{x^{n+1}}{n+1} + C \]
Table of Antidifferentiation Formula

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<td>$c \cdot F(x)$</td>
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<td>$\cos x$</td>
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27] Find all functions $g$ s.t. $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$

**Solution:**

$g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$

$\therefore g(x) = 4(-\cos x) + 2\left(\frac{2}{5} x^5\right) - \frac{1}{5} x^{\frac{5}{2}} + C$

$= -4 \cos x + \frac{2}{5} x^5 - 2x^{\frac{1}{2}} + C$
[Ex 3] Find \( f \) in \( f(x) = e^x + 20(1 + x^2)^{-1} \) and \( f(0) = -2 \).

[Sol]:

\[ f(x) = e^x + \frac{20}{1 + x^2} \]

\[ \Rightarrow f(x) = e^x + 20 \tan^{-1}x + C \]

Since \( f(0) = -2 \), we have \( f(0) = e^0 + 20 \tan^{-1}0 + C = -2 \)

\[ \Rightarrow C + 1 = -2 \quad \Rightarrow C = -3 \]

So the particular solution is \( f(x) = e^x + 20 \tan^{-1}x - 3 \).

[Ex 4] Find \( f \) if \( f''(x) = 12 x^2 + 6 x - 4 \), \( f(0) = 4 \) and \( f(1) = 1 \)

[Sol]:

\[ f(x) = 12 \left( \frac{1}{3} x^3 \right) + 6 \left( \frac{1}{2} x^2 \right) - 4 \cdot x + C = 4x^3 + 3x^2 - 4x + C \]

\[ \Rightarrow f(x) = 4 \left( \frac{1}{4} x^4 \right) + 3 \left( \frac{1}{2} x^2 \right) - 4 \left( \frac{1}{2} x^2 \right) + Cx + D = x^4 + x^2 - 2x^2 + Cx + D \]

\[ \Rightarrow f(0) = 4 \quad \Rightarrow f(0) = D = 4 \]

\[ \Rightarrow f(1) = 1 \quad \Rightarrow f(1) = 1 + 1 - 2 + C + D = 1 \quad \Rightarrow C + D = 1 \quad \Rightarrow C = -3 \]

Thus, the required function is \( f(x) = x^4 + x^2 - 2x^2 - 3x + 4 \).
EX 5] The graph of a function $f$ is given below. Make a rough sketch of an antiderivative $F$, given $F(0) = 2$.

**Solution:**

Note that $F'(x) = f(x)$

1. $f = F' < 0$ on $(0, 1) \Rightarrow F$ is decreasing on $(0, 1)$
2. $f = F' > 0$ on $(1, 3) \Rightarrow F$ is increasing on $(1, 3)$
3. $f = F' < 0$ on $(3, \infty) \Rightarrow F$ is decreasing on $(3, \infty)$

4. $F$ has a local min. at $x = 1$ (horizontal tangent)
5. $F$ has a local max. at $x = 3$ (horizontal tangent)

6. $f(x) \to 0$ as $x \to \infty \Rightarrow$ the graph of $F$ becomes flatter as $x \to \infty$.

Also notice that $F''(x) = f'(x)$

7. $f' = F'' > 0$ on $(0, 2) \Rightarrow F$ is CU on $(0, 2)$
8. $f' = F'' < 0$ on $(2, 4) \Rightarrow F$ is CD on $(2, 4)$
9. $f' = F'' > 0$ on $(4, \infty) \Rightarrow F$ is CU on $(4, \infty)$

10. $F$ has inflection points when $x = 2$ and $x = 4$
Ex 6] If \( f(x) = \sqrt{1 + x^3} - x \), sketch the graph of the antiderivative \( F \) that satisfies the initial condition \( F(-1) = 0 \).

Solu]

You may draw the graph of \( f \) first and then use it to graph \( F \) as in Ex 5.

But, this time let's create a more accurate graph by using what is called a direction field instead.

A direction field for \( f(x) = \sqrt{1 + x^3} - x \)
The slope of the line segments above \( x = a \) is \( f(a) \)

The graph of an antiderivative \( F \) satisfying \( F(-1) = 0 \) follows the direction field.
A particle moves in a straight line and has acceleration given by \( a(t) = 6t + 4 \). Its initial velocity is \( v(0) = -6 \text{ cm/s} \) and its initial displacement is \( s(0) = 9 \text{ cm} \). Find its position function \( s(t) \).

**Solution:**

\[
\begin{align*}
\vdash v'(t) &= a(t) = 6t + 4 \\
\vdash v(t) &= 3t^2 + 4t + C \\
\text{Since } v(0) &= -6, \text{ we have } v(0) = C = -6 & \vdash v(t) &= 3t^2 + 4t - 6 \\
\text{Next, } s'(t) &= v(t) = 3t^2 + 4t - 6 \\
\vdash s(t) &= t^3 + 2t^2 - 6t + D \\
\text{Since } s(0) &= 9, \text{ we have } s(0) = D = 9 \\
\text{Thus, } s(t) &= t^3 + 2t^2 - 6t + 9
\end{align*}
\]
A ball is thrown upward with a speed of 48 ft/s from the edge of a cliff 432 ft above the ground. Find its height above the ground t seconds later. When does it reach its maximum height? When does it hit the ground.

1. The motion is vertical and the height above the ground at time \( t \) is its position function \( S(t) \). We choose the positive direction to be upward. Since the velocity \( v(t) \) is decreasing, the acceleration must be negative.
   
   \[ a(t) = v'(t) = -32 \quad \Rightarrow \quad v(t) = -32t + C. \]
   
   \[ \therefore \quad v(0) = 48 \quad \Rightarrow \quad v(0) = C = 48. \quad \text{Therefore} \quad v(t) = -32t + 48. \]

   Since \( S'(t) = 0 \) \( \Rightarrow 0 = -32t + 48 \), we have \( S(t) = -16t^2 + 48t + D \)

   \[ \therefore S(0) = 432 \quad \Rightarrow \quad 432 = D. \quad \text{Thus,} \quad S(t) = -16t^2 + 48t + 432 \]

2. The ball reaches its maximum height \( \iff S'(t) = v(t) = 0 \)

   \[ -32t + 48 = 0 \quad \iff \quad t = \frac{48}{32} = \frac{3}{2} \text{ (sec.)} \]

3. The ball hits the ground \( \iff S(t) = 0 \iff -16t^2 + 48t + 432 = 0 \)

   \[ t^2 - 3t - 27 = 0 \quad \iff \quad t = \frac{3 \pm 3\sqrt{13}}{2} \]

   We reject the solution with the minus sign since \( \frac{3 - 3\sqrt{13}}{2} < 0 \)

   Therefore, the ball hits the ground after \( \frac{3 + 3\sqrt{13}}{2} \approx 6.9 \text{ sec.} \).