§ 11.1 Sequence

[EX1] Three descriptions of the sequence

(a) \( \left\{ \frac{n}{n+1} \right\}_{n=1}^{\infty} \); \( a_{n} = \frac{n}{n+1} \); \( \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots \right\} \)

(b) \( \left\{ \frac{(-1)^{n}(n+1)}{2^{n}} \right\} \); \( a_{n} = \frac{(-1)^{n}(n+1)}{2^{n}} \); \( \left\{ \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \ldots, \frac{(-1)^{n}(n+1)}{2^{n}}, \ldots \right\} \)

(c) \( \left\{ \sqrt{n-3} \right\}_{n=3}^{\infty} \); \( a_{n} = \sqrt{n-3}, n \geq 3 \); \( \left\{ 0, 1, \sqrt{5}, \sqrt{6}, \ldots, \sqrt{n-3}, \ldots \right\} \)

(d) \( \left\{ \cos \frac{n\pi}{6} \right\}_{n=0}^{\infty} \); \( a_{n} = \cos \frac{n\pi}{6}, n \geq 0 \); \( \left\{ 1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, \ldots, \cos \frac{n\pi}{6}, \ldots \right\} \)

[EX2] Find a formula for the general term \( a_{n} \) of the sequence

\( \left\{ \frac{2}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots \right\} \)

assuming that the pattern of the first terms continues.

[Sol1]:

\[ a_{1} = \frac{3}{5}, \ a_{2} = -\frac{4}{25} = -\frac{4}{5^{2}}, \ a_{3} = \frac{5}{125} = \frac{5}{5^{3}}, \ a_{4} = -\frac{6}{625} = -\frac{6}{5^{4}}, \ldots \]

\[ a_{n} = (-1)^{n+1}(n+2) \quad \frac{1}{5^{n}} \]
[Ex 3] The Fibonacci sequence \( \{ f_n \} \) is defined recursively by the conditions
\[
f_1 = 1, \quad f_2 = 1, \quad f_n = f_{n-1} + f_{n-2}, \quad n \geq 3
\]
Each term is the sum of the two preceding terms. The first few terms are
\[
\{ 1, 1, 2, 3, 5, 8, 13, 21, \ldots \}
\]

**Def.**
A sequence \( \{ a_n \} \) has the limit \( L \) and we write
\[
\lim_{n \to \infty} a_n = L \quad \text{or} \quad a_n \to L \quad \text{as} \quad n \to \infty
\]
if we can make the terms \( a_n \) as close to \( L \) as we like by taking \( n \) sufficiently large. If \( \lim_{n \to \infty} a_n \) exists, we say the sequence **converges** (or is convergent). Otherwise, we say the sequence **diverges** (or is divergent).

**Theorem**
If \( \lim_{x \to \infty} f(x) = L \) and \( f(n) = a_n \) when \( n \) is an integer, then \( \lim_{n \to \infty} a_n = L \)
[Ex 4] Find \( \lim_{n \to \infty} \frac{n}{n+1} \)

[Sol]:
\[
\lim_{x \to \infty} \frac{x}{x+1} = 1 \quad \Rightarrow \quad \lim_{n \to \infty} \frac{n}{n+1} = 1
\]

[Ex 5] Calculate \( \lim_{n \to \infty} \frac{\ln n}{n} \)

[Sol]:
\[
\lim_{x \to \infty} \frac{\ln x}{x} = \lim_{x \to \infty} \frac{1}{x} = 0 \quad \Rightarrow \quad \lim_{n \to \infty} \frac{\ln n}{n} = 0
\]

(by L'Hospital's Rule)

[Ex 6] Determine whether the sequence \( a_n = (-1)^n \) is convergent or divergent.

[Sol]:
The sequence is \{ -1, 1, -1, 1, -1, \ldots \}

Since the terms oscillate between 1 and -1 infinitely often, \( a_n \) does not approach any number. Thus, \( \lim_{n \to \infty} (-1)^n \) does not exist; that is, the sequence \( \{ (-1)^n \} \) is divergent.
If \( \lim |a_n| = 0 \), then \( \lim a_n = 0 \)

[Ex. 7] Evaluate \( \lim \frac{(-1)^n}{n} \) if it exists.

[Sol.]:
\[
\lim_{n \to \infty} \left| \frac{(-1)^n}{n} \right| = \lim_{n \to \infty} \frac{1}{n} = 0 \quad \Rightarrow \quad \lim_{n \to \infty} \frac{(-1)^n}{n} = 0
\]

The Limit Laws for functions also hold for the limits of sequences.

If \( \{a_n\} \), \( \{b_n\} \) are convergent sequences and \( c \) is a constant, then

1. \( \lim_{n \to \infty} (a_n \pm b_n) = \lim_{n \to \infty} a_n \pm \lim_{n \to \infty} b_n \)
2. \( \lim_{n \to \infty} c a_n = c \lim_{n \to \infty} a_n \)
3. \( \lim_{n \to \infty} a_n b_n = \lim_{n \to \infty} a_n \cdot \lim_{n \to \infty} b_n \)
4. \( \lim_{n \to \infty} \frac{a_n}{b_n} = \frac{\lim_{n \to \infty} a_n}{\lim_{n \to \infty} b_n} \), if \( \lim_{n \to \infty} b_n \neq 0 \)
5. \( \lim_{n \to \infty} a_n^p = \left( \lim_{n \to \infty} a_n \right)^p \), if \( p > 0 \) and \( \left( \lim_{n \to \infty} a_n \right)^p \) is defined
The Squeeze Theorem for sequences

If \( a_n \leq b_n \leq c_n \) for \( n \geq n_0 \) and \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} c_n = L \),
then \( \lim_{n \to \infty} b_n = L \)

[Ex8] Discuss the convergence of the sequence \( a_n = \frac{n!}{n^n} \), where \( n! = 1 \cdot 2 \cdot 3 \cdots n \).

[Sol]:
\[ a_1 = 1, \quad a_2 = \frac{1 \cdot 2}{2 \cdot 2}, \quad a_3 = \frac{1 \cdot 2 \cdot 3}{3 \cdot 3 \cdot 3}, \quad \cdots, \quad a_n = \frac{1 \cdot 2 \cdot 3 \cdots n}{n \cdot n \cdot n \cdots n} \]

\[ \Rightarrow a_n = \frac{1}{n} \left( \frac{2}{n} \cdot \frac{3}{n} \cdot \frac{4}{n} \cdots \frac{n}{n} \right) \leq \frac{1}{n} \]. So \( 0 < a_n \leq \frac{1}{n} \) for all \( n \).

Since \( \lim_{n \to \infty} 0 = \lim_{n \to \infty} \frac{1}{n} = 0 \), we have \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n!}{n^n} = 0 \) by the Squeeze Thm.

[Ex9] For what values of \( r \) is the sequence \( \{ r^n \} \) convergent?

[Sol]:
(i) If \( r > 0 \), \( \lim_{n \to \infty} r^n = \begin{cases} \frac{\infty}{0}, & \text{if } r > 1 \\ 0, & \text{if } 0 \leq r < 1 \end{cases} \)

(ii) If \( -1 < r < 0 \), then \( 0 < |r| < 1 \). So \( \lim_{n \to \infty} |r^n| = \lim_{n \to \infty} |r|^n = 0 \), thus \( \lim_{n \to \infty} r^n = 0 \)

(iii) If \( r \leq -1 \), then \( \lim_{n \to \infty} r^n \) is divergent since \( \lim_{n \to \infty} |r|^n \) is divergent.

Therefore, \( \{ r^n \} \) is convergent when \( -1 < r < 1 \).
The sequence \( \{r^n\} \) is convergent if \(-1 < r \leq 1\) and divergent for all other values of \(r\).

\[
\lim_{n \to \infty} r^n = \begin{cases} 
0 & \text{if } -1 < r < 1 \\
1 & \text{if } r = 1 
\end{cases}
\]

**Def**

A sequence \( \{a_n\} \) is called increasing if \( a_n < a_{n+1} \) for all \( n \geq 1 \).

A sequence \( \{a_n\} \) is called decreasing if \( a_n > a_{n+1} \) for all \( n \geq 1 \).

A sequence \( \{a_n\} \) is called monotonic if it is either increasing or decreasing.

[Ex 1.10] The sequence \( \left\{ \frac{3}{n+5} \right\} \) is decreasing since \( \frac{3}{n+5} > \frac{3}{n+6} = \frac{3}{(n+1)+5} \) for all \( n \).
[Ex 11] Show that the sequence \( a_n = \frac{n}{n^2 + 1} \) is decreasing.

[Sol 1]

We must show that \( \frac{n}{n^2 + 1} > \frac{n + 1}{(n + 1)^2 + 1} \)

\[
\Leftrightarrow n \left[ (n + 1)^2 + 1 \right] > (n + 1)(n^2 + 1)
\]

\[
\Leftrightarrow n^3 + 2n^2 + 2n > n^3 + n^2 + n + 1
\]

\[
\Leftrightarrow n^2 + n > 1
\]

Since, for \( n \geq 1, \ n^2 + n > 1 \). Therefore, \( \frac{n}{n^2 + 1} > \frac{n + 1}{(n + 1)^2 + 1} \) for \( n \geq 1 \).

That is, \( a_n > a_{n+1} \) for all \( n \). So \( \{a_n\} \) is decreasing.

[Sol 2]:

Let \( f(x) = \frac{x}{x^2 + 1} \). \( f'(x) = \frac{1 \cdot (x^2 + 1) - x \cdot 2x}{(x^2 + 1)^2} = \frac{1 - x^2}{(x^2 + 1)^2} < 0 \) for \( x > 1 \)

So \( f \) is decreasing on \([1, \infty)\). Therefore \( f(n) > f(n + 1) \) for all \( n \geq 1 \).

That is, \( \{a_n\} \) is decreasing.

**Def.**

1. A sequence \( \{a_n\} \) is bounded below if \( \exists M \) s.t. \( m \leq a_n \) for all \( n \geq 1 \)
2. A sequence \( \{a_n\} \) is bounded above if \( \exists M \) s.t. \( a_n \leq M \) for all \( n \geq 1 \)
3. If it is bounded below and above, then \( \{a_n\} \) is a bounded sequence.
Thm
Every bounded, monotonic sequence is convergent.

[Ex12] Investigate the sequence \( \{a_n\} \) defined by the recurrence relation
\[
a_1 = 2, \quad a_{n+1} = \frac{1}{2}(a_n + 6) \quad \text{for } n = 1, 2, 3, \ldots
\]

[Sol]:
(1) We use mathematical induction to show that \( \{a_n\} \) is decreasing, i.e. \( a_{n+1} < a_n \) for all \( n \).

For \( n = 1 \), \( a_2 = 4 > a_1 \). If we assume that it is true for \( n = k \), that is, \( a_{k+1} > a_k \), then
\[
a_{k+2} = \frac{1}{2}(a_{k+1} + 6) > \frac{1}{2}(a_k + 6) \Rightarrow a_{k+2} > a_k.
\]
Thus, \( a_{k+2} > a_{k+1} \).
We have deduced that \( a_{n+1} > a_n \) for \( n = k + 1 \).
Therefore, \( a_{n+1} > a_n \) for all \( n \).

(2) Next, we prove that \( \{a_n\} \) is bounded by showing that \( a_n < 6 \) for all \( n \).
For \( n = 1 \), \( a_1 = 2 < 6 \) is true. Suppose that it is true for \( n = k \). Then \( a_k < 6 \).
\[
\Rightarrow a_k + 6 < 12 \Rightarrow \frac{1}{2}(a_k + 6) < 6 . \text{ Thus } a_{k+1} < 6 . \text{ This shows, by induction, that } a_n < 6 \text{ for all } n.
\]

(3) Since \( \{a_n\} \) is increasing and bounded, it is convergent. That is, \( \lim_{n \to \infty} a_n = L \) exists.
\[
\Rightarrow L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \frac{1}{2}(a_n + 6) = \frac{1}{2}(\lim_{n \to \infty} a_n + 6) = \frac{1}{2}(L + 6)
\]
\[
\Rightarrow L = \frac{1}{2}(L + 6) \Rightarrow L = 6 . \text{ So we conclude that } \lim_{n \to \infty} a_n = 6 .\]
§11.2 Series

**Def**

Given a series \( \sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots \), let \( S_n \) denote its \( n \)th partial sum:

\[
S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n
\]

If the sequence \( \{ S_n \} \) is convergent and \( \lim_{n \to \infty} S_n = S \) exists as a real number, then the series \( \sum_{n=1}^{\infty} a_n \) is called convergent and we write

\[
a_1 + a_2 + a_3 + \cdots + a_n + \cdots = S \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = S
\]

The number \( S \) is called the sum of the series.

Otherwise, we call the series divergent.

Notice that

\[
\sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} \sum_{i=1}^{n} a_i
\]

[EX1] The geometric series is \( a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}, (a \neq 0) \)

For what values of \( r \) is the geometric series convergent?
[Sol]:

1. If \( r = 1 \), then \( S_n = a + a + \cdots + a = na \to \pm \infty \)

Since \( \lim_{n \to \infty} S_n \) does not exist, the geometric series is divergent in this case.

2. If \( r \neq 1 \), we have
\[
S_n = a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}
\]

(a) When \( -1 < r < 1 \), \( \lim_{n \to \infty} r^n = 0 \). So \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} \)

Thus, when \( |r| < 1 \) the geometric series is convergent and its sum is \( \frac{a}{1-r} \).

(b) When \( r \leq -1 \) or \( r > 1 \), the sequence \( \{r^n\} \) is divergent and so \( \lim_{n \to \infty} S_n \) D.N.E.

Therefore, the geometric series diverges when \( r \leq -1 \) or \( r > 1 \).

The geometric series \( \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots \) is convergent if \( |r| < 1 \)

and its sum is \( \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \), \( |r| < 1 \)

If \( |r| \geq 1 \), the geometric series is divergent.
[Ex 3] Is the series $\sum_{n=1}^{\infty} \frac{2^n}{3^{1-n}}$ convergent or divergent?

[Sol.]:
$$\sum_{n=1}^{\infty} \frac{2^n}{3^{1-n}} = \sum_{n=1}^{\infty} \frac{2^n \cdot 3}{3^n} = \sum_{n=1}^{\infty} \frac{4}{3^n}$$

We recognize this series as a geometric series with $a = 3$ and $r = \frac{4}{3}$.

Since $r > 1$, the series is divergent.

[Ex 4] Write the number $2.3\overline{17} = 2.317171717\ldots$ as a ratio of integers.

[Sol.]:
$$2.3171717\ldots = 2.3 + \frac{17}{10^3} + \frac{17}{10^5} + \frac{17}{10^7} + \ldots$$

$$= 2.3 + \frac{\frac{17}{10^3}}{1 - \frac{1}{10^2}} = 2.3 + \frac{17}{1000} \cdot \frac{100}{99} = \frac{233}{99} + \frac{17}{990} = \frac{1147}{495}$$

[Ex 5] Find the sum of the series $\sum_{n=0}^{\infty} x^n$, where $|x| < 1$.

[Sol.]:
This is a geometric series with $a = 1$ and $r = x$. Since $|x| < 1$, it converges and

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$
[Ex 6] Show that the series \( \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \) is convergent, and find its sum.

[Sol]:

\[
S_n = \sum_{i=1}^{n} \frac{1}{i(i+1)} = \sum_{i=1}^{n} \left( \frac{1}{i} - \frac{1}{i+1} \right) = (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \cdots + (\frac{1}{n} - \frac{1}{n+1}) = 1 - \frac{1}{n+1}
\]

So, \( \lim_{n \to \infty} S_n = \lim_{n \to \infty} (1 - \frac{1}{n+1}) = 1 - 0 = 1. \)

Therefore, the given series is convergent and its sum is 1.

[Ex 7] Show that the harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \) is divergent.

[Sol]:

\[
S_1 = 1 \quad ; \quad S_2 = 1 + \frac{1}{2}
\]

\[
S_4 = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) > 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) = 1 + \frac{2}{2}
\]

\[
S_8 = 1 + \frac{1}{2} + \left( \frac{1}{5} + \frac{1}{4} \right) + \left( \frac{1}{6} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} \right) > 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} \right) = 1 + \frac{3}{2}
\]

\[
S_{16} = 1 + \frac{1}{2} + \left( \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{5} + \cdots + \frac{1}{8} \right) + \left( \frac{1}{9} + \cdots + \frac{1}{16} \right)
\]

\[
> 1 + \frac{1}{2} + \left( \frac{1}{4} + \frac{1}{4} \right) + \left( \frac{1}{8} + \cdots + \frac{1}{8} \right) + \left( \frac{1}{16} + \cdots + \frac{1}{16} \right) = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2}
\]

In general, \( S_{2^n} > 1 + \frac{n}{2} \)

Since \( S_{2^n} \to \infty \) as \( n \to \infty \), the seq. \( \{S_n\} \) is divergent. Therefore, the harmonic series diverges.
Theorem 6
If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$

[Proof]:
$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} (S_n - S_{n-1}) = \lim_{n \to \infty} S_n - \lim_{n \to \infty} S_{n-1} = S - S = 0$$
($\because \sum_{n=1}^{\infty} a_n = \lim_{n \to \infty} S_n = S$)

Note that: $\lim_{n \to \infty} a_n = 0 \Rightarrow \sum_{n=1}^{\infty} a_n$ converges.

The Test for Divergence
If $\lim_{n \to \infty} a_n$ does not exist or if $\lim_{n \to \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

[EX 8] Show that the series $\sum_{n=1}^{\infty} \frac{n^2}{5n^2+4}$ diverges.

[Sol]:
$$\lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^2}{5n^2+4} = \frac{1}{5} \neq 0$$

So, by the Test for Divergence, the series is divergent.
Thm 8.

If \( \sum a_n \) and \( \sum b_n \) are convergent series, then so are the series 
\( \sum c a_n \) where \( c \) is a constant, \( \sum (a_n + b_n) \), and 
\( \sum (a_n - b_n) \), and

(i) \( \sum_{n=1}^{\infty} c a_n = c \sum_{n=1}^{\infty} a_n \)
(ii) \( \sum_{n=1}^{\infty} (a_n \pm b_n) = \sum_{n=1}^{\infty} a_n \pm \sum_{n=1}^{\infty} b_n \)

[Ex 9] Find the sum of the series \( \sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right) \)

[Sol]:

\[
\sum_{n=1}^{\infty} \frac{3}{n(n+1)} = 3 \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 3 \lim_{n \to \infty} \left( \frac{1}{1\cdot2} + \frac{1}{2\cdot3} + \frac{1}{3\cdot4} + \ldots + \frac{1}{n(n+1)} \right) \\
= 3 \lim_{n \to \infty} \left( 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} \right) = 3
\]

and \( \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1 \)

\[
\sum_{n=1}^{\infty} \left( \frac{3}{n(n+1)} + \frac{1}{2^n} \right) = \sum_{n=1}^{\infty} \frac{3}{n(n+1)} + \sum_{n=1}^{\infty} \frac{1}{2^n} = 3 + 1 = 4
\]

Note that: If \( \sum_{n=M}^{\infty} a_n \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) is convergent.

If \( \sum_{n=N}^{\infty} a_n \) is divergent, then \( \sum_{n=1}^{\infty} a_n \) is divergent.
§ 11.3 The Integral Test and Estimates of Sums

\[ y = \frac{1}{x^2} \]

Area = $\frac{1}{1^2}$

Area = $\frac{1}{2^2}$

Area = $\frac{1}{3^2}$

Area = $\frac{1}{4^2}$

Area = $\frac{1}{5^2}$

\[ \sum_{n=2}^{\infty} \frac{1}{n^2} \leq \int_{1}^{\infty} \frac{1}{x^2} \, dx \quad \text{and} \quad \int_{1}^{\infty} \frac{1}{x^2} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{x^2} \, dx = \cdots = 1 \quad \text{is convergent.} \]

\[ \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{1}{n^2} \leq 1 + 1 = 2 \quad \text{i.e. the partial sums } \{ S_n \} \text{ is bounded above.} \]

Since the partial sums are increasing and bounded above, the series \( \sum_{n=1}^{\infty} a_n \) is convergent.

\[ y = \frac{1}{\sqrt{x}} \]

Area = $\frac{1}{\sqrt{1}}$

Area = $\frac{1}{\sqrt{2}}$

Area = $\frac{1}{\sqrt{3}}$

Area = $\frac{1}{\sqrt{4}}$

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx \quad \text{and} \quad \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{\sqrt{x}} \, dx = \cdots = \infty \quad \text{is divergent.} \]

\[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \] is divergent.
The Integral Test

Suppose \( f \) is continuous, positive, decreasing on \([1, \infty)\) and let \( a_n = f(n) \).
Then \( \sum_{n=1}^{\infty} a_n \) is convergent \( \iff \int_1^{\infty} f(x) \, dx \) is convergent.

\[ \text{i.e.} \quad \begin{align*}
(\Rightarrow) & \quad \text{If } \int_1^{\infty} f(x) \, dx \text{ is convergent, then } \sum_{n=1}^{\infty} a_n \text{ is convergent.} \\
(\Leftarrow) & \quad \text{If } \int_1^{\infty} f(x) \, dx \text{ is divergent, then } \sum_{n=1}^{\infty} a_n \text{ is divergent.}
\end{align*} \]

Note that: \( \int_N^{\infty} f(x) \, dx \text{ is convergent } \iff \sum_{n=N}^{\infty} a_n \text{ is convergent } \iff \sum_{n=1}^{\infty} a_n \text{ is convergent.} \]

\([\text{Ex.1}]\) Test the series \( \sum_{n=1}^{\infty} \frac{1}{n^2+1} \) for convergence or divergence.

\([\text{Sol.}]\) \( \because \quad f(x) = \frac{1}{x^2+1} \) is continuous, positive, and decreasing on \([1, \infty)\).

and \( \int_1^{\infty} \frac{1}{x^2+1} \, dx = \lim_{t \to \infty} \int_1^{t} \frac{1}{x^2+1} \, dx = \lim_{t \to \infty} \tan^{-1}x \bigg|_1^t = \lim_{t \to \infty} (\tan^{-1}t - \tan^{-1}1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \)

\( \therefore \quad \int_1^{\infty} \frac{1}{x^2+1} \, dx \text{ is convergent.} \)

Thus, by the Integral Test, the series \( \sum_{n=1}^{\infty} \frac{1}{n^2+1} \) is convergent.
[Ex2] For what values of $p$ is the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ convergent?

[Sol]:

(1) If $p \leq 0$, then $\lim_{n \to \infty} \frac{1}{n^p} = \begin{cases} \infty, & \text{if } p < 0 \\ 1, & \text{if } p = 0 \end{cases}$

Since $\lim_{n \to \infty} \frac{1}{n^p} \neq 0$, the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is divergent. (when $p \leq 0$)

(2) If $p > 0$, then $f(x) = \frac{1}{x^p}$ is continuous, positive, and decreasing on $[1, \infty)$.

$\int_{1}^{\infty} \frac{1}{x^p} \, dx$ is convergent if $p > 1$ and divergent if $p \leq 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

The $p$-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$

[Ex3]

(a) The series $\sum_{n=1}^{8} \frac{1}{n^3}$ is convergent because it is a $p$-series with $p = 3 > 1$

(b) The series $\sum_{n=1}^{8} \frac{1}{3n}$ is divergent because it is a $p$-series with $p = \frac{1}{3} < 1$
[EX4] Determine whether the series \(\sum_{n=1}^{\infty} \frac{\ln n}{n}\) converges or diverges.

[Sol1]:

\(f(x) = \frac{\ln x}{x}\) is positive and continuous for \(x > 1\).

\[f'(x) = \frac{1 \cdot x - \ln x}{x^2} = \frac{1 - \ln x}{x^2}\]

\(: f'(x) < 0 \text{ for } \ln x > 1 \quad \text{i.e. } x > e.\]

\(: f(x) \text{ is decreasing when } x > e.\]

\[\int_{1}^{\infty} \frac{\ln x}{x} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{\ln x}{x} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{x}(\ln x)^2 \right]_{1}^{t} = \lim_{t \to \infty} \left( -\frac{1}{t} (\ln t)^2 - \frac{1}{2} \right) = \infty\]

Since \(\int_{1}^{\infty} \frac{\ln x}{x} \, dx\) is divergent, the series \(\sum_{n=1}^{\infty} \frac{\ln n}{n}\) is also divergent.

In general, \(\sum_{n=1}^{\infty} \frac{\int_{1}^{\infty} f(x) \, dx}{n}\) where \(f\) is conti., positive, and decreasing and \(f(\ln n) = a_n\).

\(\int_{1}^{\infty} f(x) \, dx\) converges.
Estimating the Sum of a Series

Suppose that \( \sum_{n=1}^{\infty} a_n = S \), that is, \( \lim_{n \to \infty} S_n = S \).

The remainder \( R_n \) is \( R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \).

If \( f(k) = a_k \), where \( f \) is a continuous, positive, decreasing function for \( x \geq n \).

\[
R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \leq \int_n^{\infty} f(x) \, dx
\]

\[
R_n = a_{n+1} + a_{n+2} + a_{n+3} + \cdots \geq \int_{n+1}^{\infty} f(x) \, dx
\]

Remainder Estimate for the Integral Test

Suppose \( f(k) = a_k \), where \( f \) is continuous, positive, decreasing function for \( x \geq n \) and \( \Sigma a_n \) is convergent. If \( R_n = S - S_n \), then

\[
\int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_n^{\infty} f(x) \, dx
\]
[EX 5] (a) Approximate the sum of the series $\sum \frac{1}{n^3}$ by using the sum of the first 10 terms. Estimate the error involved in this approximation.

(b) How many terms are required to ensure that the sum is accurate to within 0.0005?

[Sol]:

\[
\int_0^\infty \frac{1}{x^3} \, dx = \int_0^\infty \frac{1}{x^3} \, dx = \lim_{x \to \infty} \int_0^x 2x^2 \, dx = \lim_{x \to \infty} \frac{1}{x} - \frac{x}{2} = \frac{1}{2}\frac{1}{n^2}
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n^3} \approx S_{10} = \frac{1}{1^3} + \frac{1}{2^3} + \cdots + \frac{1}{10^3} \approx 1.1975
\]

\[
R_{10} \leq \int_0^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2} \bigg|_{n=10} = \frac{1}{200} = 0.005
\]

So the size of the error is at most 0.005.

(b) We want the value of $n$ that asserts

\[
R_n \leq \int_0^\infty \frac{1}{x^3} \, dx = \frac{1}{2n^2} < 0.0005
\]

\[
\Rightarrow n^2 > \frac{1}{0.001} = 1000
\]

\[
\Rightarrow n > \sqrt{1000} \approx 31.6
\]

We need 32 terms to ensure accuracy to within 0.0005.
\[ \int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx \Rightarrow S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S_n + R_n \leq \int_{n}^{\infty} f(x) \, dx + S_n \]

\[ \Rightarrow S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S \leq S_n + \int_{n}^{\infty} f(x) \, dx \]  \hspace{1cm} \text{(4)}

[EX 6] Use the inequality above with \( n=10 \) to estimate the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

[Sol.]:

With \( n=10 \), the inequality becomes

\[ S_{10} + \int_{10}^{\infty} \frac{1}{x^3} \, dx \leq S \leq S_{10} + \int_{10}^{\infty} \frac{1}{x^3} \, dx, \]

where \( \int_{n}^{\infty} \frac{1}{x^3} \, dx = \frac{1}{2n^2} \) from EX 5.

So,

\[ S_{10} + \frac{1}{2 \cdot 10^2} \leq S \leq S_{10} + \frac{1}{2 \cdot 11^2} \]

Using \( S_{10} \approx 1.197532 \), we get \( 1.201664 \leq S \leq 1.202532 \).

Therefore, \( S = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx \frac{1.201664 + 1.202532}{2} \) with error \( < \frac{1.202532 - 1.201664}{2} \)

\[ S = \sum_{n=1}^{\infty} \frac{1}{n^3} \approx 1.202098 \] with error \( < 0.0005 \)

The estimate \( S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S \leq S_n + \int_{n}^{\infty} f(x) \, dx \) is much better than the estimate \( S \approx S_n \)

(To make the error smaller than 0.0005 we had to use 32 terms in EX 5 but only 10 terms in EX 6.)
§11.4 The Comparison Tests

The Comparison Test

Suppose that ∑ \( a_n \) and ∑ \( b_n \) are series with positive terms.

(i) If ∑ \( b_n \) is convergent and \( a_n \leq b_n \) for all \( n \), then ∑ \( a_n \) is also convergent.

(iii) If ∑ \( b_n \) is divergent and \( a_n \geq b_n \) for all \( n \), then ∑ \( a_n \) is also divergent.

[Ex 1] Determine whether the series \( \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \) converges or diverges.

[Sol]: \[
\frac{5}{2n^2+4n+3} < \frac{5}{2n^2} \quad \text{for all } n.
\]

and \( \sum_{n=1}^{\infty} \frac{5}{2n^2} = \frac{5}{2} \sum_{n=1}^{\infty} \frac{1}{n^2} \) is convergent because \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) is a p-series with \( p=2 > 1 \)

\( \therefore \sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3} \) is convergent by part (i) of the Comparison Test.

[Ex 2] Test the series \( \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \) for convergence or divergence.

[Sol]: \[
\frac{\ln(n)}{n} > \frac{1}{n} \quad \text{for } n \geq 3 \quad \text{and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent (p-series with } p=1)\]

\( \therefore \sum_{n=1}^{\infty} \frac{\ln(n)}{n} \) is divergent.
The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If $\lim_{n \to \infty} \frac{a_n}{b_n} = c$ where $c$ is a finite number and $c \neq 0$, then either both series converges or both series diverges.

ie. $\sum_{n=1}^{\infty} a_n$ converges $\iff$ $\sum_{n=1}^{\infty} b_n$ converges.

[Proof]:

Let $m$ and $M$ be positive numbers s.t. $m < c < M$.

$\therefore \lim_{n \to \infty} \frac{a_n}{b_n} = c$ ie. $\frac{a_n}{b_n}$ is close to $c$ when $n$ is sufficiently large.

$\therefore$ there exists an integer $N$ s.t. $m < \frac{a_n}{b_n} < M$ when $n > N$

$\Rightarrow m b_n < a_n < M b_n$ when $n > N$

So $\sum b_n$ converges $\Rightarrow$ $\sum M b_n$ converges $\Rightarrow$ $\sum a_n$ converges by the Comparison Test.

$\sum b_n$ diverges $\Rightarrow$ $\sum M b_n$ diverges $\Rightarrow$ $\sum a_n$ diverges by the Comparison Test.
[Ex 3] Test the series \( \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \) for convergence or divergence.

[Sol1]:
\[
\lim_{n \to \infty} \frac{1}{2^n} = \lim_{n \to \infty} \frac{2^n}{2^n - 1} = 1 > 0
\]

and \( \sum_{n=1}^{\infty} \frac{1}{2^n} \) is a convergent geometric series.

\[\therefore \sum_{n=1}^{\infty} \frac{1}{2^n - 1} \text{ is convergent.}\]

[Ex 4] Determine whether the series \( \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \) converges or diverges.

[Sol1]:
\[
\lim_{n \to \infty} \frac{\frac{2n^2 + 3n}{\sqrt{5 + n^5}}}{\frac{2n^2}{\sqrt{n^5}}} = \lim_{n \to \infty} \frac{2n^2 + 3n}{2n^2 \sqrt{5 + n^5}} = \frac{2}{2} = 1 > 0
\]

and \( \sum_{n=1}^{\infty} \frac{2n^2}{\sqrt{n^5}} = 2 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} \) is divergent because it is a p-series with \( p = \frac{1}{3} \)

\[\therefore \text{The series } \sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}} \text{ is divergent.}\]
Estimating Sums

Suppose \( \sum a_n \) and \( \sum b_n \) are two convergent series with \( a_n \leq b_n \) for all \( n \).
Let \( \sum a_n = S \), \( \sum b_n = T \), and \( R_n \) and \( T_n \) be the remainder of \( \sum a_n \) and \( \sum b_n \), respectively.

\[
R_n = S - S_n = a_{n+1} + a_{n+2} + a_{n+3} + \\
T_n = T - T_n = b_{n+1} + b_{n+2} + b_{n+3} + \\
\]

Since \( a_n \leq b_n \) for all \( n \), we have \( R_n \leq T_n \).

[EX 5] Use the sum of the first 100 terms to approximate the sum of the series \( \sum_{n=1}^{\infty} \frac{1}{n^3+1} \).
Estimate the error involved in this approximation.

[Sol]:
1. \( \sum_{n=1}^{100} \frac{1}{n^3+1} \approx \sum_{n=1}^{100} \frac{1}{n^3} \approx 0.6864538 \)
2. \( \frac{1}{n^3+1} < \frac{1}{n^3} \) \( \therefore R_n \leq T_n \leq \int_n^{\infty} \frac{1}{x^3} \, dx = \frac{1}{2n^2} \)

With \( n=100 \), we have \( R_n \leq \frac{1}{2(100)^2} = 0.000005 \)

So \( \sum_{n=1}^{\infty} \frac{1}{n^3+1} \approx 0.6864538 \) with error less than 0.000005.
§11.5 Alternating Series

An alternating series is a series whose terms are alternately positive and negative.

ex. \[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\]

\[-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^n n}{n+1}\]

The Alternating Series Test

If the alternating series \(\sum_{n=1}^{\infty} (-1)^{n-1}b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdots\) \((b_n > 0)\)

satisfies (i) \(b_{n+1} \leq b_n\) for all \(n\)

(ii) \(\lim_{n \to \infty} b_n = 0\)

then the series is convergent.

[Ex1] The alternating harmonic series \[1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\]
satisfies (i) \(b_{n+1} < b_n\) for all \(n\)

(ii) \(\lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0\)

So the series is convergent by the Alternating Series Test.
Proof of the Alternating Series Test:

We first consider the even partial sums:

\[ S_2 = b_1 - b_2 \geq 0 \quad \text{since} \quad b_2 \leq b_1 \]

\[ S_4 = S_2 + (b_3 - b_4) \geq S_2 \quad \text{since} \quad b_4 \leq b_3 \]

In general \[ S_{2n} = S_{2n-2} + (b_{2n-1} - b_{2n}) \geq S_{2n-2} \quad \text{since} \quad b_{2n} \leq b_{2n-1} \]

Thus \[ 0 \leq S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_{2n} \leq \cdots \]

On the other hand, \[ S_{2n} = b_1 - (b_2 - b_3) - (b_4 - b_5) - \cdots - (b_{2n-2} - b_{2n-1}) - b_{2n} \]

Since \( b_i - b_{i+1} > 0 \) for all \( i \), we know that \( S_{2n} \leq b_1 \) for all \( n \).

So the sequence \( \{S_{2n}\} \) is increasing and bounded above by \( b_1 \).

It is therefore convergent by the Monotonic Sequence Thm. Let \( \lim_{n \to \infty} S_{2n} = S \)

\[ \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} (S_{2n} + b_{2n+1}) = \lim_{n \to \infty} S_{2n} + \lim_{n \to \infty} b_{2n+1} = S + 0 = S \]

\[ \lim_{n \to \infty} S_{2n+1} = \lim_{n \to \infty} S_{2n} = S \]

\[ \therefore \sum_{i=1}^{n} (-1)^{i-1} b_i = S \] is convergent.
[EX1] The alternating harmonic series \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \) satisfies

(i) \( b_{n+1} < b_n \) because \( \frac{1}{n+1} < \frac{1}{n} \) and

(ii) \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0 \). So the series is convergent by the Alternating Series Test.

[EX2] The series \( \sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1} \) is alternating but \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{3n}{4n-1} = \frac{3}{4} \neq 0 \)

So condition (ii) is not satisfied. But you cannot come to the conclusion that the series is divergent by the Alternating Series Test. To determine whether the series is divergent, we use the Test for Divergence. Since \( \lim_{n \to \infty} \frac{(-1)^n 3n}{4n-1} \) does not exist, the series is divergent by the Test for Divergence.

[EX3] Test the series \( \sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1} \) for convergence or divergence.

[Sol]:
This alternating series satisfies the second condition because \( \lim_{n \to \infty} \frac{n^2}{n^3+1} = 0 \)

To determine whether the sequence \( \frac{n^2}{n^3+1} \) is decreasing, we let \( f(x) = \frac{x^2}{x^3+1} \)

\( f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} \). \( f'(x) < 0 \) when \( x > \sqrt[3]{2} \). That is, \( f(n+1) < f(n) \) for \( n \geq 2 \).
Thus, the first condition is also satisfied for \( n \geq 2 \). Therefore, the series is convergent.
Estimating Sums

**Alternating Series Estimation Theorem**

If $S = \sum_{n=1}^{\infty} (-1)^{n-1}b_n$ is the sum of an alternating series that satisfies

1. $0 \leq b_{n+1} \leq b_n$ and
2. $\lim_{n \to \infty} b_n = 0$

then $|R_n| = |S - S_n| \leq b_{n+1}$

**[Proof]**:

$S_n < S < S_{n+1}$ or $S_{n+1} < S < S_n$

$\therefore |S - S_n| \leq |S_{n+1} - S_n| = b_{n+1}$

**[EX4]** Find the sum of the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

**[Sol]**:

$\therefore \frac{1}{(n+1)!} < \frac{1}{n!}$ and $\lim_{n \to \infty} \frac{1}{n!} = 0$ \quad \therefore \text{The alternating series is convergent.}$

To have accuracy to within 0.001, we want $|R_n| \leq b_{n+1} = \frac{1}{(n+1)!} < 0.001$

$\Rightarrow (n+1)! > 1000 \quad \Rightarrow \ n \geq 6$

$S_6 = -1 + \frac{1}{1} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$ with error $|R_6| < \frac{1}{7!} < 0.0002$

Therefore the sum $S \approx 0.368$ is correct to three decimal places.
Def
A series \( \Sigma a_n \) is called **absolutely convergent** if the series of absolute values \( \Sigma |a_n| \) is convergent.

[Ex1] The series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \cdots \) is absolutely convergent because
\[
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots
\]
is a convergent \( p \)-series with \( p = 2 \).

[Ex2] The alternating harmonic series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots \) is convergent, but it's not absolutely convergent because
\[
\sum_{n=1}^{\infty} \frac{|(-1)^{n-1}|}{n} = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots
\]
is divergent.

Def
A series \( \Sigma a_n \) is called **conditionally convergent** if it is convergent but not absolutely convergent.
Theorem 3
If a series \( \sum a_n \) is absolutely convergent, then it is convergent.

[Proof]:
\[ 0 \leq a_{n+1} |a_n| \leq 2 |a_n| \text{, } \sum |a_n| \text{ is absolutely convergent } \Rightarrow \sum |a_{n+1}| \text{ is convergent.} \]
\[ \Rightarrow \sum 2 |a_n| = 2 \sum |a_n| \text{ is also convergent } \Rightarrow \sum (a_{n+1} + |a_n|) \text{ is convergent.} \]
Therefore, \( \sum a_n = \sum (a_{n+1} + |a_n|) - \sum |a_n| \text{ is convergent.} \)

[Ex.3] Determine whether the series \( \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \) is convergent or divergent.

[Sol.]:
\[ \left| \frac{\cos n}{n^2} \right| = \left| \frac{\cos n}{n^2} \right| \leq \frac{1}{n^2} \text{ and } \sum \frac{1}{n^2} \text{ is convergent (p-series with } p=2) \]
\[ \Rightarrow \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \text{ is convergent and therefore } \sum_{n=1}^{\infty} \frac{\cos n}{n^2} \text{ is convergent by Thm 3.} \]

The Ratio Test
(i) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

(ii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = L > 1 \text{ or } L=\infty \), the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

(iii) If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the Ratio Test is inconclusive (no conclusion).
Note: If \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the Ratio Test gives no information.

\[ \text{Ex. (1)} \text{ For } \sum_{n=1}^{\infty} \frac{1}{n^2}, \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{1}{n^2} = \frac{n^2}{(n+1)^2} = 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n^2} \text{ is convergent.} \]

\[ \text{Ex. (2)} \text{ For } \sum_{n=1}^{\infty} \frac{1}{n}, \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{n+1}{n} = \lim_{n \to \infty} \frac{n}{n+1} = 1 \text{ and } \sum_{n=1}^{\infty} \frac{1}{n} \text{ is divergent.} \]

Therefore, if \( \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1 \), the series \( \sum a_n \) might converge or diverge.

In this case, the Ratio Test fails and we must use some other test.

[Ex 4] Test the series \( \sum_{m=1}^{\infty} (-1)^n \frac{n^3}{3^n} \) for absolute convergence.

\[ [S01]: \quad \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(-1)^{n+1} (n+1)^3}{3^{n+1}} \cdot \frac{3^n}{(-1)^n n^3} \right| = \lim_{n \to \infty} \frac{(n+1)^3}{3n^3} = \lim_{n \to \infty} \frac{1}{3} \left( \frac{n+1}{n} \right)^3 = \frac{1}{3} < 1 \]

\[ \therefore \text{ The given series is absolutely convergent.} \]

[Ex 5] Test the convergence of the series \( \sum_{n=1}^{\infty} \frac{n^n}{n!} \).

\[ [S01]: \quad \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} = \lim_{n \to \infty} \left( \frac{n+1}{n} \right)^n = e > 1 \quad \therefore \text{ diverges} \]

\[ [S02]: \quad \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{n^n}{n!} = \lim_{n \to \infty} \frac{n \cdot n \cdot n \cdots n}{1 \cdot 2 \cdot 3 \cdots n} = \infty \quad \therefore \text{ The series diverges.} \]
The Root Test

(i) If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L < 1 \), then the series \( \sum_{n=1}^{\infty} a_n \) is absolutely convergent.

(ii) If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = L > 1 \) or \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = \infty \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

(iii) If \( \lim_{n \to \infty} \sqrt[n]{|a_n|} = 1 \), the Root Test is inconclusive.

[Ex 6] Test the convergence of the series \( \sum_{n=1}^{\infty} \frac{(2n+3)^n}{n!} \frac{1}{n+2} \)

[Sol]:
\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{2n+3}{3n+2} = \frac{2}{3} < 1
\]
\therefore The series is convergent.
3.11.7 Strategy for Testing Series

Basic Facts

1. The geometric series \( \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots \)
   
   (a) If \( |r| < 1 \), it is convergent and \( \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \), \( |r| < 1 \)

   (b) If \( |r| \geq 1 \), it is divergent.

2. The \( p \)-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) is convergent if \( p > 1 \) and divergent if \( p \leq 1 \).

   The harmonic series \( \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots \) is divergent (\( p \)-series with \( p=1 \))

3. The alternating harmonic series \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \frac{1}{5} + \ldots \) is convergent.

The tests for Convergence and Divergence

(1) The Test for Divergence:

   If \( \lim_{n \to \infty} a_n \) does not exist or if \( \lim_{n \to \infty} a_n \neq 0 \), then the series \( \sum_{n=1}^{\infty} a_n \) is divergent.

(2) The Integral Test: \( f : \text{cont.}, \text{positive, decreasing function on } [1, \infty) \) and \( a_n = f(n) \)

   (a) If \( \int_{1}^{\infty} f(x) \, dx \) is convergent, then \( \sum_{n=1}^{\infty} a_n \) is convergent.

   (b) If \( \int_{1}^{\infty} f(x) \, dx \) is divergent, then \( \sum_{n=1}^{\infty} a_n \) is divergent.
(3) **The Comparison Test:** \( \sum a_n, \sum b_n \): series with positive terms. \( N \): an integer

(a) If \( \sum b_n \) is convergent and \( a_n \leq b_n \) for all \( n \geq N \), then \( \sum a_n \) is convergent.

(b) If \( \sum b_n \) is divergent and \( a_n \geq b_n \) for all \( n \geq N \), then \( \sum a_n \) is divergent.

(4) **The Limit Comparison Test**

\[ \sum a_n, \sum b_n \text{ : series with positive terms} \]

If \( \lim_{n \to \infty} \frac{a_n}{b_n} = c > 0 \) (\( c \) is finite), then \( \sum_{n=1}^{\infty} a_n \text{ converges} \iff \sum_{n=1}^{\infty} b_n \text{ converges} \)

(5) **The Alternating Series Test:**

If the alternating series \( \sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \ldots \) \((b_n > 0)\)
satisfies

(i) \( b_{n+1} \leq b_n \) for all \( n \geq N \)

(ii) \( \lim_{n \to \infty} b_n = 0 \)

then the series is convergent.

(6) **The Ratio Test:** \( \lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = L \)

(a) If \( L < 1 \), the series \( \sum a_n \) is absolutely convergent.

(b) If \( L > 1 \) or \( L = \infty \), the series \( \sum a_n \) is divergent.

(c) If \( L = 1 \), the Ratio Test is inconclusive.
(7) The Root Test: \[ \lim_{n \to \infty} \sqrt[n]{|a_n|} = L \]

(a) If \( L < 1 \), then \( \sum a_n \) is convergent.
(b) If \( L > 1 \) or \( L = \infty \), then \( \sum a_n \) is divergent.
(c) If \( L = 1 \), the Root Test is inconclusive.

Estimating Sums

(1) \( S \approx S_n \), \( R_n = S - S_n \)

\[ f(k) = a_k, \quad f: \text{conti. positive, decreasing for } x \geq n \text{ and } \sum a_n \text{ converges.} \]

\[ \int_{n+1}^{\infty} f(x) \, dx \leq R_n \leq \int_{n}^{\infty} f(x) \, dx \]

(2) \[ S_n + \int_{n+1}^{\infty} f(x) \, dx \leq S \leq S_n + \int_{n}^{\infty} f(x) \, dx \]

\[ \Rightarrow S \approx S_n + \frac{1}{2}(\int_{n}^{\infty} f(x) \, dx + \int_{n+1}^{\infty} f(x) \, dx) \text{ with error} < \frac{1}{2}(\int_{n}^{\infty} f(x) \, dx - \int_{n+1}^{\infty} f(x) \, dx) \]

(3) Alternating Series Estimation Theorem:
If \( S = \sum (-1)^{n-1} b_n \) is the sum of an alternating series satisfying
(i) \( 0 \leq b_{n+1} \leq b_n \) and (iii) \( \lim_{n \to \infty} b_n = 0 \),
then \( |R_n| = |S - S_n| \leq b_{n+1} \)
Strategy

(1) Check if it is a p-series or geometric series

(2) If \( \lim_{n \to \infty} a_n \neq 0 \) \( \Rightarrow \) it diverges by the Test for Divergence.

(3) If \( a_n = \) rational function or algebraic function

\[ \Rightarrow \text{compare with a p-series by keeping only the highest powers in the numerator & denominator} \]

(Use the Comparison Test or the Limit Comparison Test)

(4) Alternating Series \( \Rightarrow \) try the Alternating Series Test.

(5) If \( a_n \) involves factorials or other product \( \Rightarrow \) try the Ratio Test.

(6) If \( a_n = (bn)^n \) \( \Rightarrow \) try the Root Test

(7) If \( a_n = f(n) \) where \( \int_1^\infty f(x) \, dx \) is easily evaluated \( \Rightarrow \) try the Integral Test.

\[ \text{[Ex1]} \sum_{n=1}^{\infty} \frac{n-1}{2n+1} \]

\[ \Rightarrow \quad \lim_{n \to \infty} \frac{n-1}{2n+1} = \frac{1}{2} \neq 0 \quad \Rightarrow \text{it's divergent by the Test for Divergence.} \]

\[ \text{[Ex2]} \sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^2+4n^2+2} \]

\( a_n \) is an algebraic function \( \Rightarrow \) compare with \( \sum_{n=1}^{\infty} \frac{\sqrt{n^3}}{3n^3} \) \( \text{(a p-series)} \)
\[ \sum_{n=1}^{\infty} n e^{-n^2} \]

⇒ Use the Integral Test (\( \int_{0}^{\infty} x e^{-x^2} \, dx \) is easily evaluated)

or the Ratio Test.

\[ \sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1} \]

It's a alternating series ⇒ use the Alternating Series Test.

\[ \sum_{k=1}^{\infty} \frac{2^k}{k!} \]

An involves factorial \( k! \) ⇒ use the Ratio Test

\[ \sum_{n=1}^{\infty} \frac{1}{2 + 3^n} \]

The series has the form similar to the geometric series \( \sum_{n=1}^{\infty} \frac{1}{3^n} \)

⇒ use the Comparison Test.
§11.8 Power Series

• A **power series** is a series of the form

\[ \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \]

where \( c_n \)'s are constants called the coefficients.

If \( f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots \),

then \( D_f = \text{the set of all } x \text{ for which the series converges.} \)

**[EX]** If \( f(x) = 1 + x + x^2 + x^3 + \cdots \), then \( D_f = (-1, 1) \)

• A series of the form

\[ \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots \]

is called a **power series in** \( (x-a) \) or a **power series centered at** \( a \) or a **power series about** \( a \).

**[EX]** \( \sum_{n=1}^{\infty} (x-a)^n = 1 + (x-1) + (x-1)^2 + (x-1)^3 + \cdots \) is a power series in \( (x-1) \) or a power series centered at \( 1 \) or a power series about \( 1 \).
[Ex1] For what values of $x$ is the series $\sum_{n=0}^{\infty} \frac{n! x^n}{n!}$ convergent?

[Sol.]:
By the Ratio Test, if $x \neq 0$, $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(n+1)!}{n!} \frac{x^n}{x^{n+1}} \right| = \lim_{n \to \infty} \left| \frac{(n+1)x^n}{n x^n} \right| = \lim_{n \to \infty} \left| \frac{n+1}{n} \right| |x| = \infty$.

So the series is divergent when $x \neq 0$.

Thus, the series converges only when $x = 0$.

[Ex2] For what values of $x$ does the series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$ converge?

[Sol.]:
$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \left| \frac{(x-3)^{n+1}}{n+1} \cdot \frac{n}{(x-3)^n} \right| = \lim_{n \to \infty} \left| \frac{n}{n+1} \right| |x-3| = |x-3|$

The given series is convergent when $|x-3| < 1 \iff -1 < x-3 < 1 \iff 2 < x < 4$

and it is divergent when $|x-3| > 1 \iff x-3 > 1 \text{ or } x-3 < -1 \iff x > 4 \text{ or } x < 2$

In the case when $|x-3| = 1 \iff x = 2 \text{ or } x = 4$.

(i) when $x = 2$, the series is a alternating harmonic series $\sum \frac{(-1)^n}{n}$, which is convergent.

(ii) when $x = 4$, the series becomes the harmonic series $\sum \frac{1}{n}$ which is divergent.

So the given series is convergent for $2 \leq x < 4$. 
[EX3] Find the domain of the Bessel function of order 0 defined by

\[ J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!)^2} \]

[Sol.1]:

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(-1)^{n+1} x^{2(n+1)}}{2^{2(n+1)} [(n+1)!]^2} \cdot \frac{(-1)^n x^{2n}}{2^n (n!)^2} = \lim_{n \to \infty} \frac{x^2}{4(n+1)^2} = 0 < 1 \text{ for all } x \]

Thus, by the Ratio Test, the given series converges for all \( x \).

\[ \therefore \text{The domain of the Bessel function } J_0 \text{ is } (-\infty, \infty). \]

\[ J_0(x) = \lim_{n \to \infty} S_n(x) \]

where \( S_n = \sum_{i=1}^{n} \frac{(-1)^i x^{2i}}{2^{2i} (i!)^2} \)

\[ S_0(x) = 1 \]
\[ S_1(x) = 1 - \frac{1}{4} x^2 \]
\[ S_2(x) = 1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 \]
\[ S_3(x) = 1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \frac{1}{5304} x^6 \]
\[ S_4(x) = 1 - \frac{1}{4} x^2 + \frac{1}{64} x^4 - \frac{1}{5304} x^6 + \frac{1}{147456} x^8 \]
Theorem

For a given power series \( \sum_{n=0}^{\infty} c_n (x-a)^n \) there are only three possibilities:

(i) The series converges only when \( x=a \).

(ii) The series converges for all \( x \).

(iii) There is a positive number \( R \) such that the series converges if \( |x-a|<R \) and diverges if \( |x-a|>R \).

In (i), the radius of convergence \( R=0 \) and the interval of convergence is \([0,0]=\{0\}\).

In (ii), the radius of convergence \( R=\infty \) and the interval of convergence is \((-\infty,\infty)\).

In (iii), the interval of convergence is \((a-R,a+R)\) or \((a-R,a+R]\) or \([a-R,a+R)\) or \([a-R,a+R]\).

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[EX4] Find the radius of convergence and interval of convergence of the series
\[ \sum_{n=0}^{\infty} \frac{(-3)^n x^n}{n+1} \]

[Sol]:
\[ \lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \left| \frac{(-3)^{n+1} x^{n+1}}{n+2} \cdot \frac{n+1}{(-3)^n x^n} \right| = \lim_{n \to \infty} \left( \frac{|n+1|}{n+2} \cdot |x| \right) = 3|x| \]

\[ \therefore \text{The given series converges if } 3|x| < 1 \text{ and diverges if } 3|x| > 1 \]

\[ \therefore \text{It is convergent if } |x| < \frac{1}{3} \text{ and divergent if } |x| > \frac{1}{3} \implies R = \frac{1}{3} \]

So the series is convergent on \((-\frac{1}{3}, \frac{1}{3})\) and divergent on \((-\infty, -\frac{1}{3})\) and \((\frac{1}{3}, \infty)\)

1. When \(x = -\frac{1}{3}\),

the series becomes
\[ \sum_{n=0}^{\infty} \frac{(-3)^n \left(-\frac{1}{3}\right)^n}{n+1} = \sum_{n=0}^{\infty} \frac{1}{n+1} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \]

It is divergent because it's a p-series with \(p = 1\).

2. When \(x = +\frac{1}{3}\),

the series is
\[ \sum_{n=0}^{\infty} \frac{(-3)^n \left(+\frac{1}{3}\right)^n}{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \]

It is convergent by the Alternating Series Test.

Therefore, the given series converges when \(-\frac{1}{3} < x \leq \frac{1}{3}\). Therefore, the interval of convergence is \((-\frac{1}{3}, \frac{1}{3})\)
[Ex5] Find the radius of convergence and interval of convergence of the series

\[ \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}} \]

[Sol.]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)(x+2)^{n+1}}{3^{n+2}} \cdot \frac{3^{n+1}}{n(x+2)^n} = \lim_{n \to \infty} \left( \frac{n+1}{3n} \cdot \frac{1}{|x+2|} \right) = \frac{|x+2|}{3} \]

\[ \therefore \text{The given series is convergent if } \frac{|x+2|}{3} < 1 \text{ and divergent if } \frac{|x+2|}{3} > 1 \]

\[ \Rightarrow \text{The series converges if } |x+2| < 3 \text{ and diverges if } |x+2| > 3 \]

So the radius of convergence \( R = 3 \)

We know that the series converges when \(-3 < x+2 < 3 \Leftrightarrow -5 < x < 1\)

When \( x = -5 \), the series is \( \sum_{n=0}^{\infty} \frac{n(-3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} (-1)^n n \)

When \( x = 1 \), the series is \( \sum_{n=0}^{\infty} \frac{n(3)^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} n \)

Both series diverge because \( \lim_{n \to \infty} (-1)^n n = 0 \) and \( \lim_{n \to \infty} n = 0 \)

So the given series is convergent when \(-5 < x < 1\).

The interval of convergence is \((-5, 1)\).
§11.9 Representations of Functions as Power Series

\[ f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n , \quad |x| < 1 \]

\[ f(x) = \frac{1}{1-x} = \lim_{n \to \infty} S_n \]

where \( S_n = \sum_{i=0}^{n} x^i \)

\[ S_1 = 1 \]
\[ S_2 = 1 + x \]
\[ S_3 = 1 + x + x^2 \]
\[ \vdots \]
\[ S_8 = 1 + x + x^2 + x^3 + x^4 + \cdots + x^7 \]

[EX 1] Express \( \frac{1}{1+x^2} \) as the sum of a power series and find its interval of convergence.

[Sol]:

\[ \frac{1}{1+x^2} = \frac{1}{1-(-x^2)} = 1 - x^2 + x^4 - x^6 + x^8 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^{2n} \]

Since this is a geometric series with common ratio \( R = -x^2 \),

it is convergent when \( |x^2| < 1 \iff |x| < 1 \iff x \in (-1, 1) \)

Thus the interval of convergence is \((-1, 1)\).
[Ex2] Find a power series representation for \( \frac{1}{x+2} \)

[Sol1]:
\[
\frac{1}{x+2} = \frac{1}{2(1+ \frac{x}{2})} = \frac{1}{2} \sum_{n=0}^{\infty} \left( -\frac{x}{2} \right)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n
\]

The series converges when \( |-\frac{x}{2}| < 1 \) i.e. \( -1 < x < 2 \)

So the interval of convergence is \((-1, 2)\)

[Ex3] Find a power series representation of \( \frac{x^3}{(x+2)} \)

[Sol1]
\[
\frac{x^3}{x+2} = x^3 \cdot \frac{1}{x+2} = x^3 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} x^{n+3} = \frac{1}{2} x^3 - \frac{1}{4} x^4 + \frac{1}{8} x^5 - \frac{1}{16} x^6 + \ldots
\]

The interval of convergence of this series is \((-2, 2)\)
Theorem 2

If the power series \( \sum C_n(x-a)^n \) has radius of convergence \( R > 0 \), then the function \( f \) defined by \( f(x) = C_0 + C_1(x-a) + C_2(x-a)^2 + \cdots = \sum_{n=0}^{\infty} C_n(x-a)^n \) is differentiable (and therefore continuous) on the interval \((a-R, a+R)\) and

(i) \( f'(x) = C_1 + 2C_2(x-a) + 3C_3(x-a)^2 + 4C_4(x-a)^3 + \cdots = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1} \)

\[ \text{ie. } \frac{d}{dx} \left[ \sum_{n=0}^{\infty} C_n(x-a)^n \right] = \sum_{n=1}^{\infty} nC_n(x-a)^{n-1} = \sum_{n=1}^{\infty} nC_{n-1}(x-a)^{n-1} \]

(ii) \( \int f(x) \, dx = C + C_0(x-a) + \frac{C_1}{2}(x-a)^2 + \frac{C_2}{3}(x-a)^3 + \cdots = C + \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1} \)

\[ \text{ie } \int \left[ \sum_{n=0}^{\infty} C_n(x-a)^n \right] \, dx = \sum_{n=0}^{\infty} \frac{C_n}{n+1} \int (x-a)^{n+1} \, dx = C + \sum_{n=0}^{\infty} \frac{C_n}{n+1}(x-a)^{n+1} \]

The radii of convergence of the power series in (ii) and (iii) are both \( R \).

Notice that although the radius of convergence remains the same when a power series is differentiated or integrated, this does not mean that the interval of convergence remains the same.
[Ex4] The Bessel function \( J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n (n!)^2} \) is defined for all \( x \).

So, \( J_0(x) \) is differentiable for all \( x \) and

\[
J_0'(x) = \sum_{n=0}^{\infty} \frac{d}{dx} \frac{(-1)^n x^{2n}}{2^n (n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^n (n!)^2}
\]

[Ex5] Express \( \frac{1}{(1-x)^2} \) as a power series. What is the radius of convergence?

[Sol], Differentiating each side of the eq. \( \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n \), we get

\[
\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} n x^{n-1} = \sum_{n=0}^{\infty} (n+1) x^n
\]

The radius of convergence, by thm 2, is 1.

[Ex6] Find a power series representation for \( \ln(1-x) \) and its radius of convergence.

[Sol]:

\[
\ln(1-x) = -\int \frac{1}{1-x} \, dx = -\int (1 + x + x^2 + x^3 + \cdots) \, dx = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots) + C
\]

\[
= \sum_{n=1}^{\infty} \frac{x^n}{n} + C, \quad |x|<1.
\]

Putting \( x=0 \) in this eq., we have \( \ln 1 = 0 + C \Rightarrow C=0 \)

\[
\therefore \ln(1-x) = -(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots) = -\sum_{n=1}^{\infty} \frac{x^n}{n}, \quad |x|<1.
\]

The radius of convergence is \( 1 \).
[Ex.8] Find a power series representation for $f(x) = \tan^{-1}x$

[Sol.]

$$\tan^{-1}x = \int \frac{1}{1 + x^2} \, dx = \int (1 - x^2 + x^4 - x^6 + \ldots) \, dx = C + x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots$$

Let $x = 0 \Rightarrow \tan^{-1}0 = C \Rightarrow C = 0$.

So \( \tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \)

[Ex.8] (a) Evaluate $\int \frac{1}{1 + x^7} \, dx$ as a power series.

(b) Use (a) to approximate $\int_{0}^{\frac{1}{2}} \frac{1}{1 + x^7} \, dx$ correct to within $10^{-7}$.

[Sol.]

(a) $\int \frac{1}{1 + x^7} \, dx = \int \sum_{n=0}^{\infty} (-x^7)^n \, dx = \int \sum_{n=0}^{\infty} (-1)^n x^{7n} \, dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{7n+1}}{7n+1} = C + x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \ldots$.

The series converges for $|x^7| < 1 \Leftrightarrow |x| < 1$.

(b) $\int_{0}^{\frac{1}{2}} \frac{1}{1 + x^7} \, dx = \left( x - \frac{x^8}{8} + \frac{x^{15}}{15} - \frac{x^{22}}{22} + \ldots \right) \bigg|_{0}^{\frac{1}{2}} = \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} + \ldots$

This is an alternating series. So $S \approx S_n$ with $R_n \leq b_{n+1}$.

Set $b_{n+1} = \frac{1}{(7n+1) \cdot 2^{7n+1}} \leq 10^{-7}$ \( \Rightarrow n > 3 \). So take $n = 4$. $b_5 = \frac{1}{39 \cdot 2^{39}} \approx 6.4 \times 10^{-11}$

\[ \int_{0}^{\frac{1}{2}} \frac{1}{1 + x^7} \, dx \approx \frac{1}{2} - \frac{1}{8 \cdot 2^8} + \frac{1}{15 \cdot 2^{15}} - \frac{1}{22 \cdot 2^{22}} \approx 0.49951374 \text{ with error} \leq 6.4 \times 10^{-11} \]
§11.10 Taylor and Maclaurin Series

**Theorem 5**

If $f$ has a power series representation (expansion) at $a$, that is, if

$$f(x) = \sum_{n=0}^{\infty} C_n (x-a)^n, \quad |x-a| < R$$

then its coefficients are given by the formula

$$C_n = \frac{f^{(n)}(a)}{n!}$$

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

The series is called the **Taylor series** of the function $f$ at $a$ (or about $a$ or centered at $a$).

The Taylor Series of the function $f$ at 0, i.e.

$$f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots$$

is also called the **Maclaurin series** of the function $f$.

Note that if $f$ can be represented as a power series about $a$, then $f$ is equal to its Taylor series. But there are functions that are not equal to their Taylor series.
[Ex 1] Find the Maclaurin series of the function \( f(x) = e^x \) and its radius of convergence

[Sol]: \( f(x) = e^x \Rightarrow f^{(n)}(x) = e^x \) for all \( n \) \( \Rightarrow f^{(n)}(0) = 1 \) for all \( n \).

So the Maclaurin serie is \( \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \ldots \)

\[ \lim_{n \to \infty} \frac{a_n}{a_{n+1}} = \lim_{n \to \infty} \frac{\frac{n!}{x^n}}{\frac{(n+1)!}{x^{n+1}}} = \lim_{n \to \infty} \frac{x}{n+1} = 0 \]

\[ \therefore \text{The Maclaurin series converges for all } x \text{ and the radius of convergence is } R = \infty \]

Let \( T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!} (x-a)^i = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \ldots + \frac{f^{(n)}(a)}{n!} (x-a)^n \)

\( T_n \) is a polynomial of degree \( n \) called the \( n \)th-degree Taylor polynomial of \( f \).

If \( f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \), then \( f(x) = \lim_{n \to \infty} T_n(x) \)

For \( f(x) = e^x \),

\[ T_1(x) = 1 + x \]
\[ T_2(x) = 1 + x + \frac{x^2}{2!} \]
\[ T_3(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \]
Let \( R_n(x) = f(x) - T_n(x) \) so that \( f(x) = T_n(x) + R_n(x) \).

\( R_n(x) \) is called the \textit{remainder} of the Taylor series.

If \( \lim_{n \to \infty} R_n(x) = 0 \), then
\[
\lim_{n \to \infty} T_n(x) = \lim_{n \to \infty} [f(x) - R_n(x)] = f(x) - \lim_{n \to \infty} R_n(x) = f(x)
\]

\textbf{Theorem 8}

If \( f(x) = T_n(x) + R_n(x) \), where \( T_n \) is the \( n \)-th degree Taylor polynomial of \( f \) at \( a \) and \( \lim_{n \to \infty} R_n(x) = 0 \) for \( |x-a| < R \),

then
\[
f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \quad \text{for } |x-a| < R
\]

\textbf{Taylor's Inequality}

If \( |f^{(n+1)}(x)| \leq M \) for \( |x-a| \leq d \),

then
\[
|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \quad \text{for } |x-a| < d
\]

\[
\lim_{n \to \infty} \frac{x^n}{n!} = 0 \quad \text{for every } x \in \mathbb{R}
\]
[Ex2] Prove that $e^x$ is equal to the sum of its Maclaurin Series.

[Sol] $f(x) = e^x \Rightarrow f^{(n)}(x) = e^x$ for all $n$.

When $|x| \leq d$ (d is any positive number), $|f^{(n)}(x)| = e^x \leq e^d$

So Taylor's Inequality asserts that $|R_n| \leq \frac{e^d}{(n+1)!} |x|^{n+1}$ for $|x| \leq d$.

$\therefore \lim_{n \to \infty} \frac{e^d}{(n+1)!} |x|^{n+1} = e^d \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \Rightarrow \lim_{n \to \infty} |R_n| = 0 \Rightarrow \lim_{n \to \infty} R_n = 0$ for all $x$

Therefore, by thm 8, $e^x$ is equal to the sum of its Maclaurin series.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + $ for all $x$.

$\Rightarrow e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + $ ...

[Ex3] Find the Taylor series for $f(x) = e^x$ at $a = 2$.

[Sol]: $f^{(n)}(2) = e^2$. Its Taylor series at $a = 2$ is $\sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!} (x-2)^n = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$

As in example 1, the radius of convergence is $R = \infty$.

As in example 2, we can verify that $\lim_{n \to \infty} R_n = 0$. So $e^x = \sum_{n=0}^{\infty} \frac{e^2}{n!} (x-2)^n$.
[EX4] Find the Maclaurin series for \( \sin x \) and prove that it represents \( \sin x \) for all \( x \).

[Sol]

(1) \[ f(x) = \sin x \quad f(0) = 0 \]
\[ f'(x) = \cos x \quad f'(0) = 1 \]
\[ f''(x) = -\sin x \quad f''(0) = 0 \]
\[ f'''(x) = -\cos x \quad f'''(0) = -1 \]
\[ f^{(4)}(x) = \sin x = f(x), f^{(4)}(0) = f(0) \]

So the Maclaurin series of \( f \) is

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \ldots
\]

\[
= 0 + 1 \cdot x + 0 \cdot \frac{x^2}{2!} - \frac{1}{3!} \cdot x^3 + 0 \cdot \frac{x^4}{4!} + \ldots
\]

\[
= x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots
= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}
\]

(2) \[ |f^{(n+1)}(x)| \leq 1 \quad \text{for all } x \quad \text{by Taylor's Inequality, we have } |R_n| \leq \frac{1}{(n+1)!} \]

\[ \lim_{n \to \infty} \frac{|x|^{n+1}}{(n+1)!} = 0 \quad \text{for all } x \Rightarrow \lim_{n \to \infty} R_n = 0 \quad \text{for all } x \Rightarrow \lim_{n \to \infty} R_n(x) = 0 \quad \text{for all } x \]

So

\[
\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad \text{for all } x
\]
[Ex 5] Find the Maclaurin series for \( \cos x \).

[Sol.]
\[
\cos x = \frac{d}{dx}(\sin x) = \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)
\]
\[
= 1 - \frac{3x^2}{2!} + \frac{5x^4}{4!} - \frac{7x^6}{6!} + \ldots
\]
\[
= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots
\]

Since the Maclaurin series for \( \sin x \) converges for all \( x \), the differentiated series for \( \cos x \) also converges for all \( x \). Thus
\[
\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \text{ for all } x
\]

[Ex 6] Find the Maclaurin series for the function \( f(x) = x\cos x \).

[Sol.]
\[
x\cos x = x \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n)!} = x - \frac{x^3}{3!} + \frac{x^5}{4!} - \frac{x^7}{6!} + \ldots
\]
[Ex 7] Represent $f(x) = \sin x$ as the sum of its Taylor series centered at $\frac{\pi}{3}$.

[Sol]

(i) $f(x) = \sin x$, $f'(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

$g(x) = \cos x$, $g'(\frac{\pi}{3}) = \frac{1}{2}$

$g''(x) = -\sin x$, $g''(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$

$g'''(x) = -\cos x$, $g'''(\frac{\pi}{3}) = -\frac{1}{2}$

So, the Taylor series at $\frac{\pi}{3}$ is

$$f(\frac{\pi}{3}) + f'(\frac{\pi}{3})(x-\frac{\pi}{3}) + \frac{f''(\frac{\pi}{3})}{2!}(x-\frac{\pi}{3})^2 + \frac{f'''(\frac{\pi}{3})}{3!}(x-\frac{\pi}{3})^3 + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}(x-\frac{\pi}{3}) + \frac{\sqrt{3}}{2!(x-\frac{\pi}{3})^2} + \frac{1}{2!(x-\frac{\pi}{3})^3} + \cdots$$

$$= \frac{\sqrt{3}}{2} + \frac{1}{2}(x-\frac{\pi}{3}) - \frac{\sqrt{3}}{2!(x-\frac{\pi}{3})^2} + \frac{1}{2!(x-\frac{\pi}{3})^3} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2^{2n+1}!}(x-\frac{\pi}{3})^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}!(x-\frac{\pi}{3})^{2n+1}}$$

(2) $|f^{(n)}(x)| \leq 1 \Rightarrow |R_n(x)| \leq \frac{1}{(n+1)!}|x-\frac{\pi}{3}|^{n+1}$ for all $x$.

and $\lim_{n \to \infty} \frac{1}{(n+1)!}|x-\frac{\pi}{3}|^{n+1} = 0 \Rightarrow \lim_{n \to \infty} |R_n| = 0 \Rightarrow \lim_{n \to \infty} R_n = 0$

Therefore, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n\sqrt{3}}{2^{2n+1}!}(x-\frac{\pi}{3})^{2n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n+1}!(x-\frac{\pi}{3})^{2n+1}}$
Some important Maclaurin series:

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \quad (-1, 1)
\]

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad (-\infty, \infty)
\]

\[
\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \quad (-\infty, \infty)
\]

\[
\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \quad (-\infty, \infty)
\]

\[
\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \quad [-1, 1]
\]

[Ex 9] Evaluate \( \lim_{x \to 0} \frac{e^x - 1 - x}{x^3} \)

[Sol.]:

\[
\lim_{x \to 0} \frac{e^x - 1 - x}{x^3} = \lim_{x \to 0} \frac{(1+x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots) - 1 - x}{x^3} = \lim_{x \to 0} \frac{\frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots}{x^3}
\]

\[
= \lim_{x \to 0} \left( \frac{1}{2} + \frac{x}{3!} + \frac{x^2}{4!} + \frac{x^3}{5!} + \cdots \right) = \frac{1}{2}
\]
(EX 8) (a) Evaluate \( \int e^{-x^2} \, dx \) as an infinite series.

(b) Evaluate \( \int_0^1 e^{-x^2} \, dx \) correct to within an error of 0.001.

[Sol]:

(a) \( e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots \)

\( \therefore \int e^{-x^2} \, dx = \int (1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots + \frac{(-1)^n x^{2n}}{n!} + \cdots) \, dx \)

\( = C + x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1) \cdot n!} + \cdots \)

This series converges for all \( x \) because the series for \( e^{-x^2} \) converges for all \( x \).

(b) \( \int_0^1 e^{-x^2} \, dx = \left. \left( x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \frac{x^9}{9 \cdot 4!} - \cdots \right) \right|_0^1 = 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} - \cdots \)

To make the alternating series accurate to within 0.001, we need the value of \( n \) s.t.

\[ b_{n+1} = \frac{1}{(2n+1) \cdot n!} < 0.001 \implies n > 4. \]

Take \( n = 5 \). \( b_6 = b_6 = \frac{1}{11 \cdot 5!} = \frac{1}{1320} < 0.001 \)

So \( \int_0^1 e^{-x^2} \, dx \approx 1 - \frac{1}{3} + \frac{1}{10} - \frac{1}{42} + \frac{1}{216} \approx 0.7475 \) with error \( |R_5| \leq b_6 < 0.001 \)
Multiplication and Division of Power Series.

[Ex. 10] Find the first three nonzero terms in the Maclaurin series for (a) \( e^x \sin x \) and

(b) \( \tan x \).

[Sol]:

(a) \( e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) \left(x - \frac{x^3}{3!} + \ldots\right) \)

\[= x + x^2 + \frac{1}{3} x^3 + \ldots\]

(b) \( \tan x = \frac{\sin x}{\cos x} \)

\[= \frac{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots} \]

\[= x + \frac{1}{3} x^3 + \frac{2}{15} x^5 + \ldots\]
§11.11 The Binomial Series

The Binomial Series

If $k$ is any real number and $|x| < 1$, then

$$(1 + x)^k = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 + \ldots$$

$$= \sum_{n=0}^{\infty} \binom{k}{n} x^n$$

where $\binom{k}{n} = \frac{k(k-1) \cdots (k-n+1)}{n!}$ $(n \geq 1)$ and $\binom{k}{0} = 1$

[EX1] Expand $\frac{1}{(1+x)^2}$ as a power series.

[Sol.]

We use the binomial series with $k = -2$.

$$\binom{-2}{n} = \frac{(-2)(-3)(-4) \cdots (-2-n+1)}{n!} = \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdots (n+1)}{n!} = (-1)^n (n+1).$$

So, when $|x| < 1$,

$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} \binom{-2}{n} x^n = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \ldots$$
[EX2] Find the Maclaurin series for the function \( f(x) = \frac{1}{\sqrt{4-x}} \) and its radius of convergence.

[Sol]

\[
f(x) = \frac{1}{\sqrt{4-x}} = \frac{1}{2 \sqrt{1 - \frac{x}{4}}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}}
\]

Using the binomial series with \( k = -\frac{1}{2} \), we have the coefficient

\[
\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2}) \cdot (-\frac{3}{2}) \cdot \cdots \cdot (-\frac{1}{2} - n + 1)}{n!} = \frac{(\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2}) \cdots (-\frac{2n-1}{2})}{n!}
\]

\[
= \frac{(-1)^n \cdot 1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n \cdot n!}, \text{ when } n \geq 1, \quad \binom{-\frac{1}{2}}{0} = 1
\]

So, when \( |1 - \frac{x}{4}| < 1 \) (i.e., \( 1 < 1 < 4 \)),

\[
\frac{1}{\sqrt{4-x}} = \frac{1}{2} \left(1 - \frac{x}{4}\right)^{-\frac{1}{2}} = \frac{1}{2} \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(-\frac{x}{4}\right)^n
\]

\[
= \frac{1}{2} \left[ 1 + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(2^n \cdot n!)} \frac{x^n}{8^n} \right]
\]

\[
= \frac{1}{2} \left[ 1 + \frac{1}{8} x + \frac{1 \cdot 3}{2! \cdot 8^2} x^2 + \frac{1 \cdot 3 \cdot 5}{3! \cdot 8^3} x^3 + \cdots + \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(n!) \cdot 8^n} x^n + \cdots \right]
\]