§ 4.1 Maximum and Minimum Values

**Def**

1. (a) A function \( f \) has an **absolute maximum** (or **global maximum**) at \( x = c \) if \( f(c) \geq f(x) \) for all \( x \) in \( D_f \)
   
   The number \( f(c) \) is called the **maximum value** of \( f \) on \( D_f \)

1. (b) \( f \) has an **absolute minimum** (or **global minimum**) at \( x = c \) if \( f(c) \leq f(x) \) for all \( x \) in \( D_f \)
   
   The number \( f(c) \) is called the **minimum value** of \( f \) on \( D_f \)

The maximum and minimum values of \( f \) are called the **extreme values** of \( f \)

2. (a) A function \( f \) has a **local maximum** (or **relative maximum**) at \( x = c \) if \( f(c) \geq f(x) \) when \( x \) is near \( c \) (for all \( x \) in some open interval containing \( c \))

   The number \( f(c) \) is called the **local maximum value** of \( f \)

2. (b) \( f \) has a **local minimum** (or **relative minimum**) at \( x = c \) if \( f(c) \leq f(x) \) when \( x \) is near \( c \)

   The number \( f(c) \) is called the **local minimum value** of \( f \)
\( f(x) \) has an **absolute maximum** at \( x = e \), the absolute maximum value = \( f(e) \)

\( f(x) \) has an **absolute minimum** at \( x = a \), the absolute minimum value = \( f(a) \)

\( f(x) \) has a **local maximum** at \( x = c \), the local maximum value = \( f(c) \)

\( f(x) \) has a **local maximum** at \( x = e \), the local maximum value = \( f(e) \)

\( f(x) \) has a **local minimum** at \( x = d \), the local minimum value = \( f(d) \)

\( f(x) \) has a **local minimum** at \( x = l \), the local minimum value = \( f(l) \)

The absolute minimum is not a local minimum because it occurs at an endpoint.
[Ex1] The function $f(x) = \cos x$ takes on its (local and absolute) maximum value of 1 infinitely many times. It also takes on its (local and absolute) minimum value of -1 infinitely many times.

[Ex2] $f(x) = x^2$ has an absolute (and local) minimum value $f(0) = 0$, and it has no maximum value.

The Extreme Value Theorem (極值定理)

If $f$ is continuous on a closed interval $[a, b]$, then $f$ attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c$ and $d$ in $[a, b]$.
If \( f \) is not continuous on the closed interval \([a,b]\), it may not have extreme values. (as shown below)

\( f \) has an absolute minimum value \( = f(b) \), but no maximum value.

This continuous function \( g \) has no extreme values.

**Def**

A critical number of a function \( f \) is a number \( c \in D_f \) such that either \( f'(c) = 0 \) or \( f'(c) \) does not exist.

**[Ex]** If \( f(x) = x^3 \) then \( f'(x) = 3x^2 \). \( f'(x) = 0 \) \( \iff 3x^2 = 0 \) \( \iff x = 0 \)

\[ \therefore x = 0 \] is a critical number of \( f \)

**[Ex]** \( x = 0 \) is the critical number of \( f(x) = |x| \) since \( f'(0) \) does not exist.
[Ex7] Find the critical numbers of \( f(x) = x^3 (4-x) \)

[Sol]:

\[
\begin{align*}
    f'(x) &= \frac{3}{5} x^{-\frac{2}{5}} (4-x) + \frac{3}{5} x^{-\frac{3}{5}} (-1) \\
          &= \frac{3(4-x) - 5x}{5x^{\frac{2}{5}}} \\
          &= \frac{12 - 8x}{5x^{\frac{2}{5}}}
\end{align*}
\]

\[
f'(x) = 0 \iff 12 - 8x = 0 \iff x = \frac{3}{2}
\]

\[
f'(x) \text{ D.N.E} \iff x = 0
\]

Thus, the critical numbers are \( x = \frac{3}{2} \) and \( x = 0 \)

[Ex]

The critical numbers of \( f' \) are \( x = c, x = e, x = l \left( f'(x) = 0 \right) \) and \( x = d \left( f'(x) \text{ D.N.E} \right) \)

Note that the local maximum and minimum occur at these points.
**Thm**

If $f$ has a local maximum or minimum at $x = c$, then $c$ is a critical number of $f$.

The theorem asserts that every local maximum or minimum occur at a critical number. But, be careful! The converse 反敘述 is false!! That means it may happen that $c$ is a critical number of $f$, but $f$ has no local maximum or minimum at $x = c$.

**[Ex]** $f(x) = x^3$

$x = 0$ is a critical number of $f$ since $f'(0) = 0$,

but there's no local maximum or minimum at $x = 0$

$f'(0) = 0$ simply means that the curve $y = x^3$

has a horizontal tangent at $x = 0$.

**[Ex]** $f(x) = \frac{1}{x^3}$

$x = 0$ is a critical number of $f$ since $f'(0)$ D.N.E,

but $f$ has no local maximum or minimum at $x = 0$.

Here, "$f'(0)$ D.N.E" simply means that the curve

$y = x^{\frac{1}{3}}$ has a vertical tangent at $x = 0$. 
**The Closed Interval Method** （閉區間法）

To find the absolute maximum and minimum values of a continuous function $f$ on a closed interval $[a, b]$

1. Find the values of $f$ at the critical numbers of $f$ in $(a, b)$.
2. Find the values of $f$ at the endpoints 端點 of the interval.
3. The largest of the values from step 1 and 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

**[Ex8]** Find the absolute maximum and minimum values of the function

$$f(x) = x^3 - 3x^2 + 1, \quad -\frac{1}{2} \leq x \leq 4$$

**[Sol]**: Since $f(x)$ is continuous on the closed interval $\left[-\frac{1}{2}, 4\right]$, we can use the Closed Interval Method

1. $f'(x) = 3x^2 - 6x = 3x(x - 2), \quad f'(x) = 0 \iff x = 0$ or $x = 2 \iff$ critical numbers
   
   $f'(0) = 1, \quad f'(2) = -3$

2. $f\left(-\frac{1}{2}\right) = \frac{1}{8}, \quad f(4) = 17$

3. the absolute maximum value $= f(4) = 17$
   
   the absolute minimum value $= f(2) = -3$
[Ex9] Find the absolute maximum and minimum values of the function

\[ f(x) = x - 2 \sin x, \quad 0 \leq x \leq 2\pi \]

[Sol]:

1. \( f'(x) = 1 - 2 \cos x \),

   \[ f''(x) = 0 \iff \cos x = \frac{1}{2} \iff x = \frac{\pi}{3} \text{ or } x = \frac{5\pi}{3} \]

   \[ f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - 2 \sin \frac{\pi}{3} = \frac{\pi}{3} - \sqrt{3} < 0 \]

   \[ f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} - 2 \sin \frac{5\pi}{3} = \frac{5\pi}{3} + \sqrt{3} \approx 6.96 \]

2. \( f(0) = 0 \quad ; \quad f(2\pi) = 2\pi \approx 6.28 \)

3. the absolute maximum value = \( f\left(\frac{5\pi}{3}\right) = \frac{5\pi}{3} + \sqrt{3} \)

   the absolute minimum value = \( f\left(\frac{\pi}{3}\right) = \frac{\pi}{3} - \sqrt{3} \)
Find the absolute maximum and minimum of \( f(x) = |6 - 4x| \) on \([-3, 3]\)

**[Sol]:**

1. \( f'(x) = \begin{cases} 
-4, & \text{when } x < \frac{3}{2} \\
4, & \text{when } x > \frac{3}{2}
\end{cases} \)
   
   \( f'(\frac{3}{2}) \) D.N.E.

   \( \therefore \frac{3}{2} \) is a critical number. \( f(\frac{3}{2}) = 0 \)

2. \( f(-3) = 18, \) \( f(3) = 6 \)

3. the absolute maximum = \( f(-3) = 18 \)

   the absolute minimum = \( f(\frac{3}{2}) = 0 \)
§ 4.2 The Mean Value Theorem (平均值定理)

Rolle’s Theorem（洛爾定理）

Let $f$ be a function that satisfies the following three hypothesis:

1. $f$ is continuous on the closed interval $[a, b]$.
2. $f$ is differentiable on the open interval $(a, b)$.
3. $f(a) = f(b)$

Then there is a number $c$ in $(a, b)$ such that $f'(c) = 0$

[Proof]:

Case 1: If $f(x) = k$ (a constant), then $f'(x) = 0 \ \forall x \in (a, b)$ [Fig. 1]

Case 2: If $f(x) > f(a)$ for some $x \in (a, b)$. [Fig. 2 and 3]

Since $f$ is continuous on a closed interval, by the Extreme Value Theorem, $f$ has a maximum value Somewhere in $[a, b]$. Because $f(a) = f(b)$, the maximum value must occurs at a number $c$ in $(a, b)$. That is, $f(c)$ is a local maximum value.

Since $f$ is differentiable at $c$ by hypothesis 2, we have $f'(c) = 0$
Case 3: If \( f(x) < f(a) \) for some \( x \in (a, b) \). [Fig. 4]

Similarly, \( f' \) has a minimum value in \([a, b]\).
Since \( f(a) = f(b) \), the minimum value must occurs at a number \( c \) in \((a, b)\). And therefore \( f(c) \) is a local minimum value.
Again, \( f'(c) = 0 \) since \( f' \) is differentiable at \( c \).

[Ex1]

Let \( s = f(t) \) stand for the position function of a moving object. If the object is in the same place at two different instants \( t = a \) and \( t = b \), then \( f(a) = f(b) \).
Rolle’s Theorem says there is some instant of time \( t = c \) between \( a \) and \( b \) such that \( f'(c) = 0 \), that is, the velocity is 0. \((v(c) = 0)\)
[Ex2] Prove that the equation \( x^3 + x - 1 = 0 \) has exactly one real root.

[Sol]:

(1) Let \( f(x) = x^3 + x - 1 \). Then \( f(0) = -1 < 0 \) and \( f(1) = 1 > 0 \) 
Since \( f \) is a polynomial, \( f \) is continuous on \([0, 1]\) 
By the Intermediate Value Theorem, there is a number \( c \in (a, b) \) such that \( f(c) = 0 \). Thus, the equation has a root.

(2) To show that the equation has exactly one root, we use Rolle’s Theorem and argue by contradiction. Suppose that the equation had two roots \( a \) and \( b \), then \( f(b) = f(a) = 0 \). Besides, since \( f \) is a polynomial, it is differentiable on \((a, b)\) and continuous on \([a, b]\). Thus, by Rolle’s Theorem, there exists a number \( c \in (a, b) \) s.t. \( f'(c) = 0 \). But \( f'(x) = 3x^2 + 1 \geq 1 \) for all \( x \), so, \( f'(x) \) can never be 0.
This gives a contradiction. Therefore, the equation can’t have two real roots.
That is, it has exactly one root.
The Mean Value Theorem

Let \( f \) be a function that satisfies the following hypotheses:

1. \( f \) is continuous on the closed interval \([a, b]\)
2. \( f \) is differentiable on the open interval \((a, b)\)

Then there is a number \( c \) in \((a, b)\) such that

\[
f'(c) = \frac{f(b) - f(a)}{b - a} \quad \text{or} \quad f(b) - f(a) = f'(c)(b - a)
\]
The equation of the secant line $AB$ is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

or

$$y = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

Let $h(x) = f(x) - \left[ f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \right]$.

Since $h(x)$ is the sum of $f$ and a first-degree polynomial, both of which are continuous on $[a, b]$ and differentiable on $(a, b)$, we know that $h(x)$ is also continuous on $[a, b]$ and differentiable on $(a, b)$ and $h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a}$.

Besides, $h(a) = f(a) - \left[ f(a) + \frac{f(b) - f(a)}{b - a}(a - a) \right] = 0$ and

$$h(b) = f(b) - \left[ f(a) + \frac{f(b) - f(a)}{b - a}(b - a) \right] = 0 \quad \text{i.e.} \quad h(a) = h(b) = 0$$

Therefore, by Rolle's Theorem, there exists a number $c$ in $(a, b)$ such that

$$h'(c) = 0 \quad \text{that is,} \quad h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} = 0 \quad \text{i.e.} \quad f'(c) = \frac{f(b) - f(a)}{b - a}$$
[Ex3] Consider \( f(x) = x^3 - x \), \( a = 0 \), \( b = 2 \)

[Sol]:

Since \( f \) is a polynomial, \( f' \) is continuous on \([0, 2]\) and differentiable on \((0, 2)\). Therefore, by Mean Value Theorem, there is a number \( c \in (0, 2) \) s.t. \( f(2) - f(0) = f'(c)(2 - 0) \).

Substitute \( f'(2) = 6 \), \( f(0) = 0 \) and \( f'(x) = 3x^2 - 1 \) into the equation, we get

\[
6 - 0 = (3c^2 - 1)(2 - 0)
\]

\[
\Rightarrow 6c^2 = 8 \Rightarrow c^2 = \frac{4}{3} \Rightarrow c = \pm \frac{2}{\sqrt{3}}
\]

But \( c \) must lie in \((0,2)\), so \( c = \frac{2}{\sqrt{3}} \)

The main significance 重要性 of the Mean Value Thm is that it enables us to obtain information about a function from information about its derivative.
[Ex5] Suppose that \( f(0) = -3 \) and \( f'(x) \leq 5 \) for all values of \( x \).
How large can \( f(2) \) possibly be?

[Sol]:

Since \( f'(x) \) exists for all \( x \), that is, \( f \) is differentiable and therefore continuous everywhere. In particular, we can apply the Mean Value Theorem on the interval \([0, 2]\).
There exists a number \( c \in (0, 2) \) s.t. \( f(2) - f(0) = f'(c)(2 - 0) \).
\[
\Rightarrow f(2) = f(0) + 2f'(c) = -3 + 2f'(c) \leq -3 + 2 \cdot 5 = 7
\]
The largest possible value for \( f(2) \) is 7.
Theorem 5
If \( f'(x) = 0 \) for all \( x \) in an interval \((a, b)\), then \( f \) is constant on \((a, b)\)

[Proof]:
Let \( x_1 \) and \( x_2 \) be any two numbers in \((a, b)\) with \( x_1 < x_2 \).
Since \( f \) is differentiable and therefore continuous on \((a, b)\), it must be differentiable on \((x_1, x_2)\) and continuous on \([x_1, x_2]\). By applying the Mean Value Theorem to \( f \) on the interval \([x_1, x_2]\), we know that there is a number \( c \) such that \( f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \).
Since \( f'(c) = 0 \), we have \( f(x_2) - f(x_1) = 0 \) i.e. \( f(x_2) = f(x_1) \)
Therefore, \( f \) has the same value at any two numbers in \((a, b)\).
This means \( f \) is constant on \((a, b)\).

Corollary 7
If \( f'(x) = g'(x) \) for all \( x \) in an interval \((a, b)\), then \( f - g \) is constant on \((a, b)\); that is, \( f(x) = g(x) + C \) where \( C \) is a constant.
[Proof]:

Let $F(x) = f(x) - g(x)$, then $F'(x) = f'(x) - g'(x) = 0$ for all $x$ in $(a, b)$.
Thus, by theorem 5, we conclude that $F$ is constant. i.e. $f - g$ is constant.

[Ex6] Prove the identity $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$

[Sol]:

Let $f(x) = \tan^{-1} x + \cot^{-1} x$
then $f'(x) = \frac{1}{1 + x^2} - \frac{1}{1 + x^2} = 0$ for all $x$
Therefore, $f(x) = C$ where $C$ is a constant.
To determine the value of $C$, we substitute 1 for $x$ into the equation
$C = f(1) = \tan^{-1}(1) + \cot^{-1}(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$
Thus, $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$
§ 4.3 How Derivatives Affect the Shape of a Graph

導數如何影響圖形的形狀

Increasing / Decreasing Test (I / D Test) 遞增-遞減檢驗法

(a) If $f''(x) > 0$ on an interval, then $f$ is increasing on that interval.
(b) If $f''(x) < 0$ on an interval, then $f$ is decreasing on that interval.

[Proof]:

(1) Let $x_1$ and $x_2$ be any two numbers in the interval with $x_1 < x_2$.
Since $f$ is differentiable (and therefore continuous) on that interval,
we know $f$ is differentiable on $(x_1, x_2)$ and continuous on $[x_1, x_2]$.
So by the Mean Value Theorem, there is a number $c \in (x_1, x_2)$ s.t.

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0 \quad (\because f'(c) > 0 \text{ and } x_2 - x_1 > 0)$$

i.e. $f(x_2) > f(x_1)$

(2) Part (b) is proved similarly.
[Ex1] Find where the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is increasing and where it is decreasing?

[Sol]:

$$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$$

$$f'(x) = 0 \iff x = 0, 2, -1$$

<table>
<thead>
<tr>
<th>Interval</th>
<th>$x&lt;1$</th>
<th>$-1 &lt; x &lt; 0$</th>
<th>$0 &lt; x &lt; 2$</th>
<th>$x &gt; 2$</th>
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<tbody>
<tr>
<td>$f''(x)$</td>
<td>–</td>
<td>+</td>
<td>–</td>
<td>+</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>↓</td>
<td>↑</td>
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<td>↑</td>
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</table>

So $f$ is increasing (↗) on $(-1, 0)$ and $(2, \infty)$

and it is decreasing (↘) on $(-\infty, -1)$ and $(0, 2)$
The First Derivative Test

Suppose that $x = c$ is a critical number of a continuous function $f$.

(a) If $f'$ changes from positive to negative at $x = c$, then $f$ has a local maximum at $x = c$.

(b) If $f'$ changes from negative to positive at $x = c$, then $f$ has a local minimum at $x = c$.

(c) If $f'$ does not change sign at $x = c$, then $f$ has no local maximum or minimum at $x = c$. 
[Ex2] Find the local maximum and minimum values of the function $f$ in Ex1.

[Sol]:

(1) Since $f' < 0$ when $x < -1$ and $f' > 0$ when $-1 < x < 0$
    $f(-1) = 0$ is a local minimum value
(2) Since $f' > 0$ when $-1 < x < 0$ and $f' < 0$ when $0 < x < 2$
    $f(0) = 5$ is a local maximum value
(3) Since $f' < 0$ when $0 < x < 2$ and $f' > 0$ when $x > 2$
    $f(2) = -27$ is a local minimum value
[Ex3] Find the local maximum and minimum values of the function

\( g(x) = x + 2 \sin x \), \( 0 \leq x \leq 2\pi \)

[Sol]: \( g'(x) = 1 + 2 \cos x \)

\( g'(x) = 0 \iff \cos x = -\frac{1}{2} \iff x = \frac{2\pi}{3}, \frac{4\pi}{3} \leftarrow \text{critical numbers} \)

<table>
<thead>
<tr>
<th>Interval</th>
<th>( 0 &lt; x &lt; \frac{2\pi}{3} )</th>
<th>( \frac{2\pi}{3} &lt; x &lt; \frac{4\pi}{3} )</th>
<th>( \frac{4\pi}{3} &lt; x &lt; 2\pi )</th>
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<tbody>
<tr>
<td>( g'(x) )</td>
<td>+</td>
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<td>+</td>
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<tr>
<td>( g(x) )</td>
<td>↗</td>
<td>↘</td>
<td>↗</td>
</tr>
</tbody>
</table>

By the First Derivative Test

the local maximum value = \( g\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3} + 2 \sin \frac{2\pi}{3} = \frac{2\pi}{3} + 2\left(\frac{\sqrt{3}}{2}\right) = \frac{2\pi}{3} + \sqrt{3} \)

the local minimum value = \( g\left(\frac{4\pi}{3}\right) = \frac{4\pi}{3} + 2 \sin \frac{4\pi}{3} = \frac{4\pi}{3} + 2\left(-\frac{\sqrt{3}}{2}\right) = \frac{4\pi}{3} - \sqrt{3} \)
What Does $f''$ Say about $f$?

**Def**

1. If the graph of $f'$ lies above all of its tangent lines on an interval $I$, then it is called **concave upward** 上凹 (CU) on $I$.
2. If the graph of $f'$ lies below all of its tangent lines on an interval $I$, then it is called **concave downward** 下凹 (CD) on $I$.

**Concavity Test** 凹性検驗法

(a) If $f''(x) > 0$ for all $x$ in $I$, then the graph of $f$ is concave upward (CU) on $I$.
(b) If $f''(x) < 0$ for all $x$ in $I$, then the graph of $f$ is concave downward (CD) on $I$. 
**Def**
A point $P$ on the curve $y = f(x)$ is called an *inflection point* 反曲點 if $f$ is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at $P$.

**[Ex5]** Sketch a possible graph of a function $f$ that satisfies the following conditions

(i) $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$

(ii) $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$

(iii) $\lim_{x \to -\infty} f(x) = -2$, $\lim_{x \to \infty} f(x) = 0$

**[Sol]:**
By (i), we know that $f \nearrow$ on $(-\infty, 1)$ and $\searrow$ on $(1, \infty)$

By (ii), we know that $f$ is CU on $(-\infty, -2)$ and $(2, \infty)$ and $f$ is CD on $(-2, 2)$

By (iii), we know that $y = -2$ and $y = 0$ are horizontal asymptotes of $y = f(x)$
**The Second Derivative Test**

Suppose $f''$ is continuous near $c$

(a) If $f'(c) = 0$ and $f''(c) > 0$, then $f$ has a local minimum at $x = c$.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then $f$ has a local maximum at $x = c$.

[Ex6] Discuss the curve $y = x^4 - 4x^3$ w.r.t. concavity, points of inflection, and local maximum or minimum. Use this information to sketch the curve.

[Sol]:

If $f(x) = x^4 - 4x^3$, then $f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$

and $f''(x) = 12x^2 - 24x = 12x(x - 2)$.

Therefore, $f'(x) = 0 \Rightarrow x = 0, x = 3$ (critical numbers)

Since $f''(3) = 36 > 0$, $f(3) = -27$ is a local minimum

Since $f''(0) = 0$, the Second Derivative Test gives no information about the critical number 0.

But, by the First Derivative Test, since $f'(x) < 0$ for $x < 0$ and $0 < x < 3$, $f$ has no local maximum or minimum at 0.

Set $f''(x) = 0 \iff x = 0, x = 2$
Therefore, the inflection points are (0,0) and (2,−16)

<table>
<thead>
<tr>
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<th>$x&lt;0$</th>
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</tr>
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<tbody>
<tr>
<td>$f''(x)$</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>CU</td>
<td>CD</td>
<td>CU</td>
</tr>
</tbody>
</table>

Note: (1) The Second Derivative Test is inconclusive 沒有結論的 when $f''(c) = 0$. It gives no information about the critical number $c$ if $f''(c) = 0$.
So when $f'(c) = 0$ and $f''(c) = 0$ → Use the First Derivative Test.

(2) The Second Derivative Test fails when $f'(c)$ D.N.E → Use the First Derivative Test.
[Ex7] Sketch the graph of the function $f(x) = x^3 (6 - x)^3$

[Sol]:

$$f'(x) = -\frac{4 - x}{x^3 (6 - x)^2}$$

$$f'(x) = 0 \iff x = 4$$

$$f'(x) \text{ D.N.E } \iff x = 0, x = 6$$

$\therefore x = 0, 4, 6$ are critical numbers

<table>
<thead>
<tr>
<th>$x&lt;0$</th>
<th>$0&lt;x&lt;4$</th>
<th>$4&lt;x&lt;6$</th>
<th>$x&gt;6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f'(x)$</td>
<td>–</td>
<td>+</td>
<td>–</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>↓</td>
<td>↑</td>
<td>↓</td>
</tr>
</tbody>
</table>

$\therefore f(0) = 0$ is a local minimum

$$f(4) = 2^\frac{5}{3}$$ is a local maximum

$f(x)$ has no local maximum or minimum at $x = 6$. 

$f''(x) = -\frac{8}{x^3 (6 - x)^3}$

$f''(x)$ D.N.E $\iff x = 0, x = 6$

<table>
<thead>
<tr>
<th>$x&lt;0$</th>
<th>$0&lt;x&lt;6$</th>
<th>$x&gt;6$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(x)$</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>CD</td>
<td>CD</td>
</tr>
</tbody>
</table>

$\therefore$ the point of inflection is $(6,0)$
[Ex8] Use the first and second derivative of \( f(x) = e^{\frac{1}{x}} \), together with asymptotes, to sketch its graph.

[Sol]:

(1) \( f'(x) = -\frac{e^x}{x^2} \)

\( f'(x) \) D.N.E \( \iff \) \( x = 0 \)

\[
\begin{array}{c|cc}
  f'(x) & x<0 & x>0 \\
  \downarrow & - & - \\
  f(x) & \downarrow & \downarrow \\
\end{array}
\]

\( \therefore \) \( f \) has no local maximum or minimum

(2) \( f''(x) = \frac{e^x(2x+1)}{x^4} \)

\( f''(x) = 0 \iff x = -\frac{1}{2} \)

\( f''(x) \) D.N.E \( \iff x = 0 \)

\[
\begin{array}{c|ccc}
  f''(x) & x < -\frac{1}{2} & -\frac{1}{2} < x < 0 & x > 0 \\
  & - & + & + \\
  f(x) & CD & CU & CU \\
\end{array}
\]

\( \therefore \) the inflection point is \( \left( -\frac{1}{2}, e^{-2} \right) \)
(3) \[ \lim_{x \to 0^+} e^x = \infty \quad \therefore x = 0 \text{ is a vertical asymptote} \]

\[
\left\{
\begin{array}{c}
\lim_{x \to 0^-} e^x = 0 \\
\end{array}
\right\}
\]

\[ \lim_{x \to \pm\infty} e^{-x} = e^0 = 1 \quad \therefore y = 1 \text{ is a horizontal asymptote} \]
§ 4.4 Indeterminate Forms and L’Hospital’s Rule
不定型與洛必達法則

If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \), then the limit \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is called an indeterminate form of type \( \frac{0}{0} \).

If \( \lim_{x \to a} f'(x) = \infty \) (or \( -\infty \)) and \( \lim_{x \to a} g(x) = \infty \) (or \( -\infty \)), then the limit \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is called an indeterminate form of type \( \frac{\infty}{\infty} \).

**L’Hospital’s Rule**

Suppose \( f \) and \( g \) are differentiable and \( g'(x) \neq 0 \) near \( a \) (except possibly at \( a \)).

Suppose that \( \lim_{x \to a} \frac{f(x)}{g(x)} \) is an indeterminate form of type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \).

Then \( \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)} \) if the limit \( \lim_{x \to a} \frac{f'(x)}{g'(x)} \) exist (or is \( \infty \) or \( -\infty \)).
Vote 1: It is especially important to verify the conditions regarding the limits of \( f \) and \( g \) before using L’Hospital’s Rule.

Vote 2: L’Hospital’s Rule also valid for one-sided limit and for limits at infinity or negative infinity; that is, "\( x \to a \)" can be replace by \( x \to a^+ \), \( x \to a^- \), \( x \to \infty \) or \( x \to -\infty \).

[Ex1] Find \( \lim_{x \to 1} \frac{\ln x}{x - 1} \)

[Sol]:

The limit is an indeterminate form of type \( \frac{0}{0} \), we can apply L’Hospital’s Rule:

\[
\lim_{x \to 1} \frac{\ln x}{x - 1} = \lim_{x \to 1} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x - 1)} = \lim_{x \to 1} \frac{1}{1} = 1
\]
[Ex2] Calculate \( \lim_{x \to \infty} \frac{e^x}{x^2} \)

[Sol]:

The limit is an indeterminate form of type \( \frac{\infty}{\infty} \), we can apply L’Hospital’s Rule:

\[
\lim_{x \to \infty} \frac{e^x}{x^2} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(x^2)} = \lim_{x \to \infty} \frac{e^x}{2x} = \lim_{x \to \infty} \frac{\frac{d}{dx}(e^x)}{\frac{d}{dx}(2x)} = \lim_{x \to \infty} \frac{e^x}{2} = \infty
\]

\( \lim_{x \to \infty} \frac{e^x}{2x} \) is still an indeterminate form of type \( \frac{\infty}{\infty} \)

so use the L’Hospital’s Rule again

[Ex3] Calculate \( \lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} \)

[Sol]: Apply L’Hospital’s Rule to it because it’s an indeterminate form of type \( \frac{\infty}{\infty} \)

\[
\lim_{x \to \infty} \frac{\ln x}{\sqrt[3]{x}} = \lim_{x \to \infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(\sqrt[3]{x})} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{3x^{\frac{2}{3}}}} = \lim_{x \to \infty} \frac{3}{x^{\frac{2}{3}}} = 0
\]
[Ex4] Find \( \lim_{x \to 0} \frac{\tan x - x}{x^3} \)

[Sol]:
It’s of the type \( \frac{0}{0} \), so we can apply L’Hospital’s Rule:

\[
\lim_{x \to 0} \frac{\tan x - x}{x^3} = \lim_{x \to 0} \frac{\sec^2 x - 1}{3x^2} = \lim_{x \to 0} \frac{2\sec x \cdot \sec x \tan x}{6x} = \frac{1}{3} \lim_{x \to 0} \sec^2 x \cdot \frac{\sin x}{\cos x} \cdot \frac{1}{x}
\]

\[
= \frac{1}{3} \lim_{x \to 0} \sec^2 x \cdot \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \frac{1}{3} \cdot 1 \cdot 1 \cdot 1 = \frac{1}{3}
\]

[Ex5] Find \( \lim_{x \to \infty} \frac{x}{x + \sin x} \)

[Sol]:

\[
\lim_{x \to \infty} \frac{x}{x + \sin x} \neq \lim_{x \to \infty} \frac{1}{1 + \cos x}
\]

the limit does not exist.

\[
\lim_{x \to \infty} \frac{x}{x + \sin x} = \lim_{x \to \infty} \frac{x}{x + \sin x} = \lim_{x \to \infty} \frac{1}{1 + \frac{\sin x}{x}} = 1
\]
**Indeterminate Products 不定乗積**

If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = \infty \) (or \( -\infty \)), then the limit \( \lim_{x \to a} f(x)g(x) \) is called an indeterminate form of type \( 0 \cdot \infty \).

\[
\lim_{x \to a} f(x)g(x) = \begin{cases} 
\lim_{x \to a} \frac{f(x)}{g(x)} & \text{← indeterminate form of type } \frac{0}{0} \\
\lim_{x \to a} \frac{g(x)}{f(x)} & \text{← indeterminate form of type } \frac{\infty}{\infty}
\end{cases}
\]

**[Ex6] Evaluate** \( \lim_{x \to 0^+} x \cdot \ln x \)

**[Sol]:**

Since \( \lim_{x \to 0^+} x = 0 \) and \( \lim_{x \to 0^+} \ln x = -\infty \), the limit is an indeterminate form of type \( 0 \cdot \infty \).

Using L’Hôpital’s Rule, (but converting the limit into the form of type \( \frac{0}{0} \) or \( \frac{\infty}{\infty} \) first), we have

\[
\lim_{x \to 0^+} x \cdot \ln x = \lim_{x \to 0^+} \frac{\ln x}{1/x} = \lim_{x \to 0^+} \frac{d}{dx} \left( \frac{1}{x} \right) = -\lim_{x \to 0^+} \frac{1}{x^2} = \lim_{x \to \infty} -\frac{1}{x^2} = 0
\]
If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = \infty \), then the limit \( \lim_{x \to a} [f(x) - g(x)] \) is called an indeterminate form of type \( \infty - \infty \).

In this case, we try to convert the difference into a quotient so that we have an indeterminate form of type \( \frac{0}{0} \) or \( \infty \).

**[Ex7] Compute** \( \lim_{x \to \left(\frac{\pi}{2}\right)^-} (\sec x - \tan x) \)

**[Sol]:**

This is an indeterminate form of \( \infty - \infty \). We’ll try to convert the difference into a quotient.

\[
\lim_{x \to \left(\frac{\pi}{2}\right)^-} (\sec x - \tan x) = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \left( \frac{1}{\cos x} - \frac{\sin x}{\cos x} \right) = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{1 - \sin x}{\cos x} = \lim_{x \to \left(\frac{\pi}{2}\right)^-} \frac{-\cos x}{-\sin x} = 0
\]

(indeterminate form of type \( \frac{0}{0} \))
**Indeterminate Powers**

If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \), then the limit \( \lim_{x \to a} [f(x)]^{g(x)} \) is called an **indeterminate form of type** \( 0^0 \).

If \( \lim_{x \to a} f(x) = \infty \) and \( \lim_{x \to a} g(x) = 0 \), then the limit \( \lim_{x \to a} [f(x)]^{g(x)} \) is called an **indeterminate form of type** \( \infty^0 \).

If \( \lim_{x \to a} f(x) = 1 \) and \( \lim_{x \to a} g(x) = \pm\infty \), then the limit \( \lim_{x \to a} [f(x)]^{g(x)} \) is called an **indeterminate form of type** \( 1^\infty \).

In these cases, we’ll write the function \( [f(x)]^{g(x)} \) as an exponential:

\[
[f(x)]^{g(x)} = e^{\ln(f(x))^{g(x)}} = e^{g(x)\ln f(x)}
\]

**[Ex9]** Find \( \lim_{x \to 0^+} x^x \)

**[Sol]:** It’s an indeterminate form of type \( 0^0 \)

\[
\lim_{x \to 0^+} x^x = \lim_{x \to 0^+} e^{\ln x^x} = \lim_{x \to 0^+} e^{x\ln x} = e^{\lim_{x \to 0^+} x\ln x} = e^0 = 1
\]

(\( \because e^x \) is continuous and \( \lim_{x \to 0} x \cdot \ln x = 0 \) exists by ex6)
[Ex8] Calculate \( \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} \)

[Sol]:

This is an indeterminate form of type \(1^\infty\)

\[
\lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} e^{\ln(1 + \sin 4x)^{\cot x}} = \lim_{x \to 0^+} e^{\cot x \cdot \ln(1 + \sin 4x)}
\]

Since \( \lim_{x \to 0^+} \cot x \cdot \ln(1 + \sin 4x) = \lim_{x \to 0^+} \frac{\ln(1 + \sin 4x)}{\tan x} = \lim_{x \to 0^+} \frac{\cos 4x \cdot 4}{1 + \sin 4x} = 4 \)

we have that \( \lim_{x \to 0^+} (1 + \sin 4x)^{\cot x} = \lim_{x \to 0^+} e^{\cot x \cdot \ln(1 + \sin 4x)} \)

\[
= e^{\lim_{x \to 0^+} \cot x \cdot \ln(1 + \sin 4x)} = e^{4} \quad (\because e^x \text{ is a continuous function})
\]
§ 4.5 Summary of Curve Sketching

Guidelines for Sketching a Curve: (描繪曲線的方法）

A. **Domain** (定義域)

B. **Intercepts** (截距)

C. **Symmetry**: (i) \( f(-x) = f(x) \) \( \iff \) \( f \) is an **even function**

\( \iff \) the graph of \( f \) is symmetric about the \( y \)-axis.

(ii) \( f(-x) = -f(x) \) \( \iff \) \( f \) is an **odd function**

\( \iff \) the graph of \( f \) is symmetric about the origin.

(iii) \( f(x + p) = f(x) \) for all \( x \in D_f \), where \( p \) is a positive integer

\( \iff \) \( f \) is a **periodic function** (週期函數)

D. **Asymptotes**: find **vertical asymptotes or horizontal asymptotes or slant asymptotes**

\( \iff \) the line \( y = ax + b \) is called a **slant asymptote**.

[Ex] If \( f(x) = x + \frac{1}{x} \), then \( \lim_{x \to \infty} \left[ f(x) - x \right] = \lim_{x \to \infty} \frac{1}{x} = 0 \).

Therefore \( y = x \) is a slant asymptote of \( y = f(x) \)
E. **Intervals of Increase or Decrease** (遞增或遞減區間)

F. **Local Maximum and Minimum Values** (局部極大或極小值)

G. **Concavity and Points of Inflection** (凹性與反曲點)

H. **Sketch the Curve**

[Ex1] Sketch the curve \( y = \frac{2x^2}{x^2 - 1} \)

[Sol]:

Let \( f(x) = \frac{2x^2}{x^2 - 1} \)

A. \( D_f = \{ x | x \neq \pm 1 \} = (-\infty, -1) \cup (-1, 1) \cup (1, \infty) \)

B. The \( x \)-intercept = 0 and the \( y \)-intercept = 0

C. **Symmetry**

Since \( f(-x) = f(x) \), \( f \) is even. The curve is symmetric about the \( y \)-axis.

D. **Asymptotes**

\[
\lim_{x \to 1^+} f(x) = \infty, \quad \lim_{x \to 1^-} f(x) = -\infty, \quad \lim_{x \to -1^+} f(x) = -\infty, \quad \lim_{x \to -1^-} f(x) = \infty
\]

\[
\therefore x = 1 \text{ and } x = -1 \text{ are vertical asymptotes of the curve } y = \frac{2x^2}{x^2 - 1}
\]

\[
\lim_{x \to \pm \infty} f(x) = 2 \quad \therefore y = 2 \text{ is the horizontal asymptotes of the curve.}
\]
E. Intervals of Increase or Decrease

\[ f'(x) = \frac{4x(x^2 - 1) - 2x^2 \cdot 2x}{(x^2 - 1)^2} = \frac{-4x}{(x^2 - 1)^2} \], \quad f'(x) = 0 \iff x = 0 \text{ (critical number)}

<table>
<thead>
<tr>
<th>Interval</th>
<th>( x &lt; 0, x \neq -1 )</th>
<th>( x &gt; 0, x \neq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f'(x) )</td>
<td>+</td>
<td>–</td>
</tr>
<tr>
<td>( f(x) )</td>
<td>↑</td>
<td>↓</td>
</tr>
</tbody>
</table>

F. The local maximum value = \( f(0) = 0 \)

G. Concavity and Points of Inflection

\[ f''(x) = \frac{-4(x^2 - 1)^2 - (-4x) \cdot 2(x^2 - 1) \cdot 2x}{(x^2 - 1)^4} = \frac{12x^2 + 4}{(x^2 - 1)^3} \]

\( f''(x) \) D.N.E. \iff \( x^2 - 1 = 0 \iff x = \pm 1 \)
There is no inflection point because $x = \pm 1$ are not in the domain of $f$.

<table>
<thead>
<tr>
<th></th>
<th>$x &lt; -1$</th>
<th>$-1 &lt; x &lt; 1$</th>
<th>$x &gt; 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(x)$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>CU</td>
<td>CD</td>
<td>CU</td>
</tr>
</tbody>
</table>

\[ y = \frac{2x^2}{x^2 - 1} \]
[Ex2] Sketch the graph of \( f(x) = 5(x-1)^{\frac{2}{3}} - 2(x-1)^{\frac{5}{3}} \)

[Sol]:

A. \( D_f = \Re \)

B. Intercepts:

\[ y = 5(x-1)^{\frac{2}{3}} - 2(x-1)^{\frac{5}{3}} \quad \text{when } x = 0, \ y = 7 \]
\[ \quad \text{when } y = 0, \ (x-1)^{\frac{2}{3}} [5 - 2(x-1)] = 0 \Rightarrow x = 1 \text{ or } \frac{7}{2} \]

C. Symmetry: None

D. Asymptote: None

E. Intervals of Increase or Decrease

\[ f''(x) = \frac{10(2-x)}{3(x-1)^{\frac{4}{3}}}, \ f''(x) = 0 \Leftrightarrow x = 2 \]

\[ f''(x) \ D.N.E \Leftrightarrow x = 1. \]

\[
\begin{array}{c|c|c|c}
& x<1 & 1<x<2 & x>2 \\
f'(x) & + & - & - \\
f(x) & \text{CU} & \text{CD} & \text{CU} \\
\end{array}
\]

F. The local maximum value = \( f(2) = 3 \)
   The local minimum value = \( f(1) = 0 \)
G. Concavity and Points of Inflection

\[ f''(x) = \frac{10(1-2x)}{9(x-1)^3}, \quad f''(x) = 0 \iff x = \frac{1}{2} \]

\[ f''(x) \text{ D.N.E. } \iff x = 1 \]

<table>
<thead>
<tr>
<th></th>
<th>(-\infty &lt; x &lt; \frac{1}{2})</th>
<th>(\frac{1}{2} &lt; x &lt; 1)</th>
<th>(x &gt; 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(f''(x))</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>(f(x))</td>
<td>CU</td>
<td>CD</td>
<td>CD</td>
</tr>
</tbody>
</table>

\[ \therefore \text{ the inflection point is } \left( \frac{1}{2}, 3\sqrt{2} \right) \]
[Ex3] Sketch the graph of \( f(x) = xe^x \)

[Sol]:
A. \( D_f = \mathbb{R} \)
B. The \( x \)-intercept and \( y \)-intercept are both 0.
C. Symmetry: None
D. Asymptotes:
\[
\lim_{x \to \infty} xe^x = \infty \quad \lim_{x \to -\infty} xe^x = 0
\]
\[\therefore y = 0 \text{ is the horizontal asymptote}\]
E. Intervals of Increase or Decrease
\[
f'(x) = xe^x + e^x = (x + 1)e^x
\]
\[
f'(x) = 0 \iff x = -1
\]
\[
\begin{array}{c|cc}
 & -\infty < x < -1 & -1 < x < \infty \\
f'(x) & - & + \\
f(x) & \downarrow & \uparrow
\end{array}
\]
F. The local min. value \( f(-1) = -\frac{1}{e} \)
There’s no local maximum

G. \( f''(x) = (x + 1)e^x + e^x = (x + 2)e^x \)
\[
f''(x) = 0 \iff x = -2
\]

\[
\begin{array}{c|c|c}
 & -\infty < x < -2 & -2 < x < \infty \\
f''(x) & - & + \\
f(x) & \text{CD} & \text{CU}
\end{array}
\]
\[\therefore \text{the inflection point is } (-2, -2e^{-2})\]
[Ex4] Sketch the graph of \( f(x) = 2\cos x + \sin 2x \)

[Sol]:

A. \( D_f = \mathbb{R} \)

B. the \( y \)-intercepts is \( f(0) = 2 \)
   
   the \( x \)-intercepts: \( 2\cos x + \sin 2x = 0 \Rightarrow 2\cos x(1 + \sin x) = 0 \)
   
   \( \Rightarrow \cos x = 0 \) or \( \sin x = -1 \)
   
   \( \Rightarrow x = \frac{\pi}{2} \) or \( x = \frac{3\pi}{2} \) (in \([0, 2\pi]\))

C. \( f \) is neither odd nor even, but \( f(x + 2\pi) = f(x) \) for all \( x \). Therefore \( f \) is a periodic function with period \( 2\pi \). We may consider only \( 0 \leq x \leq 2\pi \).

D. Asymptote: None

E. \( f''(x) = -2\sin x + 2\cos 2x = -2\sin x + 2(1 - 2\sin^2 x) \)

\[ = -2(2\sin^2 x + \sin x - 1) = -2(\sin x + 1)(2\sin x - 1) \]

\( f''(x) = 0 \Leftrightarrow \sin x = -1 \) or \( \sin x = \frac{1}{2} \)

\( \Rightarrow \) in \([0, 2\pi]\), \( x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2} \)
\[
\begin{array}{|c|c|c|c|c|}
\hline
& \quad \quad \quad -\infty < x < \frac{\pi}{6} & \quad \quad \quad \frac{\pi}{6} < x < \frac{5\pi}{6} & \quad \quad \quad \frac{5\pi}{6} < x < \frac{3\pi}{2} & \quad \quad \quad \frac{3\pi}{2} < x < \infty \\
\hline
f'(x) & + & - & + & + \\
\hline
f(x) & \uparrow & \downarrow & \uparrow & \uparrow \\
\hline
\end{array}
\]

\[f'(x) = -2 \cos x (1 + 4 \sin x)\]

\[f''(x) = 0 \iff \cos x = 0 \text{ or } \sin x = -\frac{1}{4} \Rightarrow x = \frac{\pi}{2}, \frac{3\pi}{2}, \alpha_1, \alpha_2\]

where \(\alpha_1 = \pi + \sin^{-1} \left( \frac{1}{4} \right)\)

\(\alpha_2 = 2\pi - \sin^{-1} \left( \frac{1}{4} \right)\)
H. We draw the curve on $[0, 2\pi]$ first, then extend the curve by translation.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$0 &lt; x &lt; \frac{\pi}{2}$</th>
<th>$\frac{\pi}{2} &lt; x &lt; \alpha_1$</th>
<th>$\alpha_1 &lt; x &lt; \frac{3\pi}{2}$</th>
<th>$\frac{3\pi}{2} &lt; x &lt; \alpha_2$</th>
<th>$\alpha_2 &lt; x &lt; 2\pi$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f''(x)$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>$f(x)$</td>
<td>CD</td>
<td>CU</td>
<td>CD</td>
<td>CU</td>
<td>CD</td>
</tr>
</tbody>
</table>

The inflection points are $(\frac{\pi}{2}, 0), (\alpha_1, f(\alpha_1)), (\frac{3\pi}{2}, 0), (\alpha_2, f(\alpha_2))$.
[Ex5] Sketch the graph of $y = \ln(4 - x^2)$

[Sol]:

Let $f(x) = \ln(4 - x^2)$

A. $D_f = \{ x \mid 4 - x^2 > 0 \} = \{ x \mid -2 < x < 2 \} = (-2, 2)$

B. the $y$-intercepts is $f'(0) = \ln 4$

the $x$-intercepts: $\ln(4 - x^2) = 0 \Rightarrow 4 - x^2 = 1 \Rightarrow x = \pm \sqrt{3}$

C. Symmetry:

$\therefore f(-x) = f(x) \therefore f$ is an even function

The curve is symmetric about $y$-axis.

D. Asymptote:

$\lim_{x \to 2^-} \ln(4 - x^2) = -\infty$ ; $\lim_{x \to 2^+} \ln(4 - x^2) = -\infty$

$\therefore x = 2$ and $x = -2$ are vertical asymptotes.

E. Intervals of Increase or Decrease

\[
f'(x) = \frac{-2x}{4 - x^2}
\]

\[
f''(x) = 0 \iff x = 0
\]

\[
\begin{array}{c|c|c}
& -2<x<0 & 0<x<2 \\
f'(x) & + & - \\
f''(x) & \uparrow & \downarrow \\
f(x) & & \\
\end{array}
\]
F. The local maximum value is $f(0) = \ln 4$.

G. Concavity and points of inflection

$$f''(x) = \frac{-2(4-x^2) - (-2x)(-2x)}{(4-x^2)^2} = \frac{-8-2x^2}{(4-x^2)^2}$$

Since $f''(x) < 0$ for all $x$ in $(-2, 2)$.

The curve is CD on $(-2, 2)$ and there is no point of inflection.

H.
[Ex6] Sketch the graph of \( f(x) = \frac{x^3}{x^2 + 1} \)

[Sol]:

A. \( D_f = \mathbb{R} \)

B. The \( x \)-intercept and \( y \)-intercept are both 0.

C. Symmetry:

Since \( f(-x) = -f(x) \), \( f \) is odd and its graph is symmetric about the origin.

D. Asymptotes:

\[
\therefore f(x) = x - \frac{x}{x^2 + 1} \quad \therefore \lim_{x \to \pm\infty} \left( f(x) - x \right) = \lim_{x \to \pm\infty} \left( -\frac{x}{x^2 + 1} \right) = 0
\]

\[\therefore y = x \text{ is a slant asymptote.}\]

E. Intervals of Increase or Decrease

\[
f'(x) = \frac{x^2(x^2 + 3)}{(x^2 + 1)^2}
\]

Since \( f''(x) > 0 \) for all \( x \in \mathbb{R}, x \neq 0 \). \( f \) is increasing on \( \mathbb{R} \).

F. There’s no local maximum or minimum
G. Concavity and points of inflection

\[
f''(x) = \frac{2x(3-x^2)}{(x^2+1)^3}, \quad f''(x) = 0 \Leftrightarrow x = 0, \ x = \pm \sqrt{3}
\]

<table>
<thead>
<tr>
<th>( -\infty &lt; x &lt; -\sqrt{3} )</th>
<th>( -\sqrt{3} &lt; x &lt; 0 )</th>
<th>( 0 &lt; x &lt; \sqrt{3} )</th>
<th>( \sqrt{3} &lt; x &lt; \infty )</th>
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<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>CU</td>
<td>CD</td>
<td>CU</td>
</tr>
</tbody>
</table>

\[ \therefore \] the inflection points are \( \left(-\sqrt{3}, -\frac{3\sqrt{3}}{4}\right), \ (0, 0), \ \left(\sqrt{3}, \frac{3\sqrt{3}}{4}\right) \)

H.
§ 4.7 Optimization Problems （最佳化問題）

[Ex1] A farmer has 2400 ft of fencing（圍繞）and wants to fence off a rectangular（矩形）field that borders（毗鄰）a straight river. He needs no fence along the river. What are the dimensions（尺寸, 寬度）of the field that has the largest area?

[Sol]:

Let $x$ and $y$ be the depth and width of the rectangle (in feet).

Then $A = xy$ and $2x + y = 2400 \Rightarrow y = 2400 - 2x$

$\Rightarrow A = x(2400 - 2x) = 2400x - 2x^2 , 0 \leq x \leq 1200$

We want to maximize $A$ which is continuous on the closed interval $[0,1200]$

$A'(x) = 2400 - 4x , A'(x) = 0 \Leftrightarrow x = 600$ (critical number) $\Rightarrow A(600) = 720000$

$\therefore A(0) = 0 , A(1200) = 0$

$\therefore A(600) = 720000$ is the absolute maximum value of the area.

OR

$\therefore A' > 0$ when $x < 600$ and $A' < 0$ when $x > 600$.

$\therefore A \nearrow$ on $(0,600)$ and $\searrow$ on $(600,1200)$

Therefore $A(600)$ is a local maximum value and also an absolute maximum value.

Thus, the rectangular field should be 600ft deep and 1200ft wide.
[Ex2] A cylindrical can (圆柱形罐头) is to be made to hold 1L of oil. Find the dimensions (尺寸) that will minimize the cost of the metal to manufacture the can.

[Sol]:
Suppose the can has radius $r$ and height $h$ (in centimeter 單位公分). In order to minimize the cost of the metal, we minimize the total surface area (表面面積) of the can.

$$A = 2\pi r^2 + 2\pi rh$$

The volume is given to be 1L ($= 1000cm^3$). Thus, $\pi r^2 h = 1000$

$$\Rightarrow h = \frac{1000}{\pi r^2}.$$ Substitution of this into the expression for $A$ gives

$$A = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r}, \quad r > 0.$$ To minimize $A$, we have to find the critical number first:

$$A' = 4\pi r - \frac{2000}{r^2} = \frac{4(\pi r^3 - 500)}{r^2}.$$ So $A' = 0 \Rightarrow r = \sqrt[3]{\frac{500}{\pi}}$

Since $A'(r) < 0$ when $r < \sqrt[3]{\frac{500}{\pi}}$ and $A'(r) > 0$ when $r > \sqrt[3]{\frac{500}{\pi}}$, we know that $A(r) \downarrow$ on $\left(0, \sqrt[3]{\frac{500}{\pi}}\right)$ and $\uparrow$ on $\left(\sqrt[3]{\frac{500}{\pi}}, \infty\right)$. 
Therefore, $A$ has an absolute minimum at $r = \frac{\sqrt[3]{500}}{\pi}$.

The volume of $h$ corresponding to $r = \frac{\sqrt[3]{500}}{\pi}$ is

$$h = \frac{1000}{\pi \left(\frac{\sqrt[3]{500}}{\pi}\right)^2} = 2 \frac{\sqrt[3]{500}}{\pi} = 2r \text{ (the diameter)}$$

Thus, the radius should be $\frac{\sqrt[3]{500}}{\pi}$ cm and the height should be equal to the diameter.
Find the point on the parabola $y^2 = 2x$ that is closest to the point $\left(1, 4\right)$.  

**[Sol]:** The distance between the point $\left(1, 4\right)$ and the point $\left(x, y\right)$ is 

$$d = \sqrt{(x-1)^2 + (y-4)^2}$$

Since $(x, y)$ lies on the parabola, we have $y^2 = 2x \Rightarrow x = \frac{y^2}{2}$

$$\Rightarrow d = \sqrt{\left(\frac{1}{2}y^2 - 1\right)^2 + (y-4)^2} = \sqrt{\frac{1}{4}y^4 - 8y + 17}$$

Instead of minimizing $d$, we minimize $d^2 = \frac{1}{4}y^4 - 8y + 17$ 

Let $f(y) = \frac{1}{4}y^4 - 8y + 17 \Rightarrow f''(y) = y^3 - 8$, so $f''(y) = 0 \iff y = 2$.

Observe that $f' < 0$ when $y < 2$ and $f' > 0$ when $y > 2$, that is, $f'$ on $(-\infty, 2)$ and $f'$ on $(2, \infty)$. Therefore, $f'$ has an absolute minimum at $y=2$.

The distance $d$ also has an absolute minimum at $y=2$. When $y = 2$, $x = \frac{y^2}{2} = 2$.

Thus, the point on $y^2 = 2x$ closest to $(1, 4)$ is $(2, 2)$. 
A man launches his boat from point $A$ on a bank of a straight river, 3km wide, and wants to reach point $B$, 8km downstream on the opposite bank, as quickly as possible. He could row his boat directly across the river to point $C$ and then run to $B$, or he could row directly to $B$, or he could row to some point $D$ between $C$ and $B$ and then run to $B$. If he can row 6km/h and run 8km/h, where should he land to reach $B$ as soon as possible? (We assume that the speed of the water is negligible compared with the speed at which the man rows.)

Let $x$ be the distance from $C$ to $D$, then the running distance is $|DB| = 8 - x$, and the rowing distance is $|AD| = \sqrt{x^2 + 9}$.

So the total time $T(x) = \frac{\sqrt{x^2 + 9}}{6} + \frac{8 - x}{8}$, $0 \leq x \leq 8$.

$$T'(x) = \frac{x}{6\sqrt{x^2 + 9}} - \frac{1}{8}$$

So $T'(x) = 0 \iff \frac{x}{6\sqrt{x^2 + 9}} = \frac{1}{8} \iff 4x = 3\sqrt{x^2 + 9} \iff 7x^2 = 81 \iff x = \frac{9}{\sqrt{7}}$ in $[0,8]$.

To find the point where the absolute minimum occur at, we compare the value of $T$ at the critical number and the end points 0 and 8.
Therefore, the absolute minimum of \( T \) on the closed interval \([0,8]\) occur at \( x = \frac{9}{\sqrt{7}} \). Thus, the man should land the boat at a point \( \frac{9}{\sqrt{7}} \) km downstream from his starting point.
Find the area of the largest rectangle that can be inscribed in a semicircle of radius $r$.

[Sol 1]:

Let $(x, y)$ be the vertex that lies in the first quadrant. Then the rectangle has sides of lengths $2x$ and $y$. So the area $A = 2xy$.

$\therefore (x, y)$ lies on the circle $x^2 + y^2 = r^2 \therefore y = \sqrt{r^2 - x^2}$

$\Rightarrow A = 2x\sqrt{r^2 - x^2} \quad 0 \leq x \leq 5$

$$A' = 2\sqrt{r^2 - x^2} - \frac{2x^2}{\sqrt{r^2 - x^2}} = \frac{2(r^2 - 2x^2)}{\sqrt{r^2 - x^2}} \quad \text{so} \quad A'(x) = 0 \Leftrightarrow x = \frac{r}{\sqrt{2}}$$

Since $A(0) = 0$, $A(r) = 0$ and $A\left(\frac{r}{\sqrt{2}}\right) = r^2$, we conclude that $A\left(\frac{r}{\sqrt{2}}\right) = r^2$ is the absolute maximum of $A$.

The area of the largest inscribed rectangle is $r^2$. 
[Sol 2]:

Let $\theta$ be the angle shown in the figure on the left. Then the area of the rectangle is

$$A(\theta) = (2r \cos \theta) \cdot r \sin \theta = r^2 \sin 2\theta.$$  

We know that $\sin 2\theta$ has a maximum value of 1 and it occurs when $2\theta = \frac{\pi}{2}$. Thus, $A(\theta)$ has a maximum value of $r^2$ and it occur when $\theta = \frac{\pi}{4}$. 
§ 4.8 (4.9) Newton’s Method (牛頓法)

To approximate a solution to the equation $f(x) = 0$, choose an initial approximation $x_1$, and calculate $x_2, x_3, x_4, \ldots$ using

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{for } n = 1, 2, 3, \ldots$$

If the numbers $x_1, x_2, x_3, \ldots$ converge, they converge to a solution of $f(x) = 0$. 
Note that $x_{n+1}$ might be a worse approximation then $x_n$ (such as $x_3$ in Fig.1) when $f''(x_n)$ is closed to 0. Then Newton’s Method fails and a better initial approximation $x_1$ should be chosen. (So does when the case in Fig.2 happens)

Newton’s Method also fails when $f'(x_n) = 0$ for some $n$.

In this case,
there is no $x_3$ produced.
[Ex] Use Newton’s Method to find the root to the equation $x^3 + 3x + 1 = 0$ to seven decimal places.

[Sol]:

Let $f(x) = x^3 + 3x + 1 = 0$, then $f'(x) = 3x^2 + 3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + 3x_n + 1}{3x_n^2 + 3}$$

The graph of $f$ suggests that choose $x_1 = -0.3$, then

$$x_2 = x_1 - \frac{x_1^3 + 3x_1 + 1}{3x_1^2 + 3} \approx -0.3223241$$

$$x_3 = x_2 - \frac{x_2^3 + 3x_2 + 1}{3x_2^2 + 3} \approx -0.3221853$$

$$x_4 = x_3 - \frac{x_3^3 + 3x_3 + 1}{3x_3^2 + 3} \approx -0.3221853$$

Since $x_3$ and $x_4$ agree to seven decimal places, we conclude that the root to $x^3 + 3x + 1 = 0$ is about $-0.3221853$. 
Usually you don’t have the graph of $f$ ready to help you decide the value of the initial approximation. In this case, you can make use of the Intermediate Value Thm: Since $f(-1) \cdot f(0) < 0$, there is a root in the interval $(-1, 0)$. Thus, you can choose $x_1 = -0.5$ to be the initial approximation. It’s also a good start.

**[Ex2]** Use Newton’s Method to find $\sqrt[6]{2}$ correct to eight decimal places.

**[Sol]**: $\sqrt[6]{2}$ is the root of the equation $x^6 - 2 = 0$.

Let $f(x) = x^6 - 2$, then $f'(x) = 6x^5$ and $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^6 - 2}{6x_n^5}$

If we choose $x_1 = 1$ as the initial approximation, then

$x_2 \approx 1.16666667$
$x_3 \approx 1.12644368$
$x_4 \approx 1.12249707$
$x_5 \approx 1.12246205$
$x_6 \approx 1.12246205$

Since $x_5$ and $x_6$ agree to eight decimal places, we conclude that $\sqrt[6]{2} \approx 1.12246205$. 
[Ex3] Find, correct to six decimal places, the root of the equation $\cos x = x$.

[Sol]:

Let $f(x) = \cos x - x$, then $f'(x) = -\sin x - 1$ and 

$$x_{n+1} = x_n - \frac{\cos x_n - x_n}{-\sin x_n - 1} = x_n + \frac{\cos x_n - x_n}{\sin x_n + 1}$$

If we choose $x_1 = 1$, then

- $x_2 \approx 0.750363$
- $x_3 \approx 0.739112$
- $x_4 \approx 0.739085$
- $x_5 \approx 0.739085$

Since $x_4$ and $x_5$ agree to six decimal places, we conclude that the root to this equation is about 0.739085.
§ 4.9(4.10) Antiderivatives (反導數)

**Def**
A function $F$ is called an antiderivative of $f$ on an interval $I$ if $F''(x) = f(x)$ for all $x$ in $I$.

Ex. $F(x) = \frac{1}{3}x^3$ and $G(x) = \frac{1}{3}x^3 + 5$ are both antiderivative of $f(x) = x^2$

**Thm 1**
If $F$ is an antiderivative of $f$ on an interval $I$, then the most general antiderivative of $f$ on $I$ is $F(x) + C$, where $C$ is an arbitrary constant.

Ex. The most general antiderivative of $f(x) = x^2$ is $\frac{1}{3}x^3 + C$
[Ex1] Find the most general antiderivative of each of the following functions.

(a) \( f(x) = \sin x \)  
(b) \( f(x) = \frac{1}{x} \)  
(c) \( f(x) = x^n, \ n \neq -1. \)

[Sol]:

(a) \( \therefore \frac{d}{dx}(-\cos x) = \sin x \quad \therefore \text{the most general antiderivative is} \ -\cos x + C \)

(b) \( \therefore \frac{d}{dx}(\ln x) = \frac{1}{x} \text{ on } (0, \infty) \)

So on the interval \((0, \infty)\), the most general antiderivative of \( f \) is \( \ln x + C \)

Also \( \frac{d}{dx}(\ln|x|) = \frac{1}{x} \) for all \( x \neq 0 \).

\( \therefore \text{on } (-\infty, 0) \text{ and } (0, \infty) \), the most general antiderivative of \( f = \frac{1}{x} \) is \( \ln|x| + C \)

Thus, the general antiderivative of \( f \) is \( F(x) = \begin{cases} 
\ln x + C & \text{if } x > 0 \\
\ln(-x) + C & \text{if } x < 0 
\end{cases} \)

(c) \( \therefore \text{when } n \neq -1, \quad \frac{d}{dx}\left(\frac{1}{n+1} x^{n+1}\right) = x^n \)

\( \therefore \text{the most general antiderivative of } f(x) = x^n \text{ is } F(x) = \frac{x^{n+1}}{n+1} + C \)
<table>
<thead>
<tr>
<th>Function</th>
<th>Particular Antiderivative</th>
<th>Function</th>
<th>Particular Antiderivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c f(x)$</td>
<td>$c F(x)$</td>
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<td>$-\cos x$</td>
</tr>
<tr>
<td>$f(x) + g(x)$</td>
<td>$F(x) + G(x)$</td>
<td>$\sec^2 x$</td>
<td>$\tan x$</td>
</tr>
<tr>
<td>$x^n (n \neq -1)$</td>
<td>$\frac{1}{n+1} x^{n+1}$</td>
<td>$\sec x \tan x$</td>
<td>$\sec x$</td>
</tr>
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<td>$\frac{1}{x}$</td>
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<td>x</td>
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<tr>
<td>$e^x$</td>
<td>$e^x$</td>
<td>$\frac{1}{1+x^2}$</td>
<td>$\tan^{-1} x$</td>
</tr>
<tr>
<td>$\cos x$</td>
<td>$\sin x$</td>
<td>$\frac{1}{x\sqrt{x^2-1}}$</td>
<td>$\sec^{-1} x$</td>
</tr>
</tbody>
</table>
[Ex2] Find all functions $g$ such that $g'(x) = 4\sin x + \frac{2x^5 - \sqrt{x}}{x}$

[Sol]:

$$g'(x) = 4\sin x + 2x^4 - x^{-\frac{1}{2}}$$

$$\therefore g(x) = 4(-\cos x) + 2\left(\frac{1}{5}x^5\right) - \frac{1}{1 \cdot 2}x^{-\frac{1}{2}} + C = -4\cos x + \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + C$$

[Ex3] Find $f$ if $f'(x) = e^x + 20\left(1 + x^2\right)^{-1}$ and $f(0) = -2$.

[Sol]:

$$\therefore f'(x) = e^x + \frac{20}{1 + x^2}$$

$$\therefore f(x) = e^x + 20\tan^{-1} x + C$$

Since $f'(0) = -2$, we have $f(0) = e^0 + 20\tan^{-1} 0 + C = -2$

$$\Rightarrow C + 1 = -2 \quad \Rightarrow C = -3$$

So the particular solution is $f(x) = e^x + 20\tan^{-1} x - 3$
[Ex4] Find $f$ if $f''(x) = 12x^2 + 6x - 4$, $f(0) = 4$ and $f(1) = 1$

[Sol]:

$$f'(x) = 12 \left( \frac{1}{3} x^3 \right) + 6 \left( \frac{1}{2} x^2 \right) - 4x + C = 4x^3 + 3x^2 - 4x + C$$

$$\Rightarrow f(x) = 4 \left( \frac{1}{4} x^4 \right) + 3 \left( \frac{1}{3} x^3 \right) - 4 \left( \frac{1}{2} x^2 \right) + Cx + D = x^4 + x^3 - 2x^2 + Cx + D$$

$\therefore f(0) = 4 \quad : \quad f(0) = D = 4$

$\therefore f(1) = 1 \quad : \quad f(1) = 1 + 1 - 2 + C + D = 1 \Rightarrow C + D = 1 \Rightarrow C = -3$

Thus, the required function is $f(x) = x^4 + x^3 - 2x^2 - 3x + 4$. 
The graph of a function $f$ is given below. Make a rough sketch of an antiderivative $F$, given $F(0) = 2$.

[Sol]:

Note that $F'(x) = f'(x)$

\[ \therefore (1) \ f = F' < 0 \text{ on } (0,1) \Rightarrow F \downarrow \text{ on } (0,1) \]

\[ (2) \ f = F' > 0 \text{ on } (1,3) \Rightarrow F \uparrow \text{ on } (1,3) \]

\[ (3) \ f = F' < 0 \text{ on } (3,\infty) \Rightarrow F \downarrow \text{ on } (3,\infty) \]

(4) $F$ has a local minimum at $x = 1$ (horizontal tangent)

(5) $F$ has a local maximum at $x = 3$ (horizontal tangent)

(6) $f(x) \to 0$ as $x \to \infty \Rightarrow$ the graph of $F$ become flatter as $x \to \infty$

Also notice that $F''(x) = f''(x)$

\[ \therefore (7) \ f' = F'' > 0 \text{ on } (0,2) \Rightarrow F \text{ is CU on } (0,2) \]

\[ (8) \ f' = F'' < 0 \text{ on } (2,4) \Rightarrow F \text{ is CD on } (2,4) \]

\[ (9) \ f' = F'' > 0 \text{ on } (4,\infty) \Rightarrow F \text{ is CU on } (4,\infty) \]

(10) $F$ has inflection points when $x = 2$ and $x = 4$. 
[Ex6] If $f(x) = \sqrt{1 + x^3} - x$, sketch the graph of the antiderivative $F$ that satisfies the initial condition $F(-1) = 0$.

[Sol]: You may draw the graph of $f$ first and then use it to graph $F$ as in Ex5. But, this time let’s create a more accurate graph by using what is called a direction field instead.

- A direction field for $f(x) = \sqrt{1 + x^3} - x$
- The slope of the line segments above $x = a$ is $f(a)$
- The graph of an antiderivative $F$ satisfying $F(-1) = 0$ follows the direction field.
**Rectilinear Motion** (直線運動)

<table>
<thead>
<tr>
<th>$s(t)$</th>
<th>differentiation</th>
<th>$v(t) = s'(t)$</th>
<th>differentiation</th>
<th>$a(t) = v'(t) = s''(t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>position function</td>
<td>antidifferentiation</td>
<td>velocity function</td>
<td>antidifferentiation</td>
<td>acceleration function</td>
</tr>
</tbody>
</table>

[Ex7] A particle 質點 moves in a straight line and has acceleration given by $a(t) = 6t + 4$.

It is initial velocity is $v(0) = -6 \text{ cm/s}$ and its initial displacement is $s(0) = 9 \text{ cm}$.

Find its position function $s(t)$.

**[Sol]:**

\[ \therefore v'(t) = a(t) = 6t + 4 \quad \therefore v(t) = 3t^2 + 4t + C \]

Since $v(0) = -6$, we have $v(0) = C = -6 \quad \therefore v(t) = 3t^2 + 4t - 6$

Next, \[ \therefore s'(t) = v(t) = 3t^2 + 4t - 6 \]

\[ \therefore s(t) = t^3 + 2t^2 - 6t + D \]

Since $s(0) = 9$, we have $s(0) = D = 9$

Thus \[ s(t) = t^3 + 2t^2 - 6t + 9 \]
[Ex8] A ball is thrown upward with a speed of 48 ft/s from the edge of a diff 432 ft above the ground. Find its height above the ground \( t \) seconds later. When does it reach its maximum height? When does it hit the ground?

[Sol]:

(1) The motion is vertical and the height above the ground at time \( t \) is its position function \( s(t) \). We choose the position direction to be upward. Since the velocity \( v(t) \) is decreasing, the acceleration must be negative.

\[
a(t) = v'(t) = -32 \quad \Rightarrow v(t) = -32t + C
\]

\[
\therefore v(0) = 48 \quad \therefore v(0) = C = 48
\]

Therefore \( v(t) = -32t + 48 \).

Since \( s'(t) = v(t) = -32t + 48 \), we have \( s(t) = -16t^2 + 48t + D \)

\[
\therefore s(0) = 432 \quad \therefore s(0) = D = 432
\]

Thus, \( s(t) = -16t^2 + 48t + 432 \).

(2) The ball reaches its maximum height \( \Leftrightarrow s'(t) = v(t) = 0 \)

\[
\Leftrightarrow -32t + 48 = 0 \quad \Leftrightarrow t = \frac{48}{32} = \frac{3}{2} \text{ (sec)}
\]
(3) The ball hit the ground $\iff s(t) = 0 \iff -16t^2 + 48t + 432 = 0$

$\iff t^2 - 3t - 27 = 0 \iff t = \frac{3 \pm 3\sqrt{13}}{2}$

We reject the solution with the minus sign since $\frac{3 - 3\sqrt{13}}{2} < 0$

Therefore, the ball hits the ground after $\frac{3 + 3\sqrt{13}}{2} \approx 6.9$ sec