§ 6.1 Areas between Curves  曲線間的面積

Formula 1:

\[ A = \int_{a}^{b} f(x) - g(x) \, dx \]

[Ex1] Find the area of the region bounded above by \( y = e^x \), bounded below by \( y = x \), and bounded on sides by \( x = 0 \) and \( x = 1 \).

[Sol]:

\[
A = \int_{0}^{1} e^x - x \, dx = \left( e^x - \frac{1}{2} x^2 \right) \bigg|_{0}^{1} = e - \frac{3}{2}
\]
[Ex2] Find the area of the region enclosed by the parabolas $y = x^2$ and $y = 2x - x^2$

[Sol]:

$$A = \int_{0}^{1} (2x - x^2) - (x^2)\,dx = \ldots = \frac{1}{3}$$

[Ex5] Find the area of the region bounded by the curves $y = \sin x$, $y = \cos x$, $x = 0$ and $x = \frac{\pi}{2}$

[Sol]:

$$\begin{cases} y = \sin x \\ y = \cos x \end{cases} \Rightarrow \sin x = \cos x \Rightarrow \tan x = 1. \quad (0 \leq x \leq \frac{\pi}{2}) \Rightarrow x = \frac{\pi}{4}$$

$$\Rightarrow A = \int_{0}^{\frac{\pi}{4}} (\cos x) - (\sin x)\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin x) - (\cos x)\,dx$$

$$= \ldots$$

$$= 2\sqrt{2} - 2$$
[Ex6] Find the area enclosed by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$

[Sol]:

\[
\begin{cases}
  y = x - 1 & \Rightarrow y^2 = 2(y + 1) + 6 \Rightarrow y^2 - 2y - 8 = 0 \\
  y^2 = 2x + 6
\end{cases}
\]

\[\Rightarrow (y - 4)(y + 2) = 0 \Rightarrow \begin{cases} y = 4 \Rightarrow x = 5 \\ y = -2 \Rightarrow x = -1 \end{cases}\]

\[
A = \int_{-1}^{-3} (\sqrt{2x + 6}) - (-\sqrt{2x + 6}) \, dx + \int_{-1}^{5} (\sqrt{2x + 6}) - (x - 1) \, dx
\]

\[
= \int_{-3}^{-1} (2x + 6)^{\frac{1}{2}} \cdot 2 \, dx + \int_{-1}^{5} \frac{1}{2} (2x + 6)^{\frac{1}{2}} \cdot 2 - x + 1 \, dx
\]

\[
= \frac{2}{3} (2x + 6)^{\frac{3}{2}} \bigg|_{-3}^{-1} + \left[ \frac{1}{2} \cdot \frac{2}{3} (2x + 6)^{\frac{3}{2}} - \frac{1}{2} x^2 + x \right]_{-1}^{5}
\]

\[= 18\]
Formula 2:

\[ A = \int_{c}^{d} f(y) - g(y) \, dy \]

[Ex] Use formula 2 to solve Ex6 again.

[Sol]:

\[
\begin{align*}
  y &= x - 1 \\
  y^2 &= 2x + 6 \\
  \Rightarrow & \quad \ldots \quad \Rightarrow y = -2, 4
\end{align*}
\]

\[
A = \int_{-2}^{4} (y + 1) - \left( \frac{1}{2} y^2 - 3 \right) \, dy
\]

\[
= \left[ -\frac{1}{6} y^3 + \frac{1}{2} y^2 + 4y \right]_{-2}^{4}
\]

\[= 18\]
Let’s divide $S$ (the solid 立體) into $n$ “slabs” 厚片 of equal width $\Delta x = \frac{b-a}{n}$ by using the planes 平面 $P_{x_1}, P_{x_2}, \ldots, P_{x_{n-1}}$ to slice 切 the solid 立體.

If we choose sample points $x_i^*$ in $[x_{i-1}, x_i]$, we can approximate the $i$th slab $S_i$ by a cylinder 柱狀體 with base 底 area $A(x_i^*)$ and height $\Delta x$.

So, $V(S_i) \approx A(x_i^*)\Delta x$

and then $V \approx \sum_{i=1}^{n} A(x_i^*)\Delta x$

we define $V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*)\Delta x$
Definition of Volume 體積的定義

Let $S$ be a solid that lies between $x = a$ and $x = b$. If the cross-sectional area of $S$ in the plane $P_x$, through $x$ and perpendicular to the $x$-axis, is $A(x)$, then the volume of $S$ is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i^*) \Delta x = \int_{a}^{b} A(x) \, dx$$
[Ex1] Show that the volume of a sphere  of radius  is  

\[ V = \frac{4}{3} \pi r^3 \]

[Sol]:

\[ \therefore y = \sqrt{r^2 - x^2} \]

\[ \therefore A(x) = \pi y^2 = \pi (r^2 - x^2) \quad (\because \pi (r^2 - x^2) \text{ is even}) \]

\[
V = \int_{-r}^{r} A(x) \, dx = \int_{-r}^{r} \pi (r^2 - x^2) \, dx = 2\pi \int_{0}^{r} r^2 - x^2 \, dx \\
= 2\pi (r^2 x - \frac{1}{3} x^3) \bigg|_{0}^{r} = 2\pi (r^3 - \frac{1}{3} r^3) \\
= \frac{4}{3} \pi r^3 \approx 4.18879 \]
A solid with a circular base of radius 1 is shown on the right. Parallel cross-sections perpendicular to the base are equilateral triangles. Find the volume of the solid.

[Sol]:

\[ A(x) = \text{the area of } \Delta ABC = \frac{1}{2} \cdot (2y) \cdot \sqrt{3}y = \sqrt{3}y^2 = \sqrt{3}(1 - x^2) \]

\[ \therefore V = \int_{-1}^{1} A(x) \, dx = \int_{-1}^{1} \sqrt{3}(1 - x^2) \, dx = 2\sqrt{3} \int_{0}^{1} 1 - x^2 \, dx = 2\sqrt{3} \left( x - \frac{1}{3} x^3 \right) \bigg|_{0}^{1} = \frac{4\sqrt{3}}{3} \]

\( (\therefore (1 - x^2) \text{ is even}) \)
[Ex8] Find the volume of a pyramid whose base is a square with side $L$ and whose height is $h$.

[Sol]:

\[ \frac{S}{L} = \frac{x}{h} \Rightarrow S = \frac{x}{h} L \]

\[ \Rightarrow A(x) = S^2 = \left(\frac{x}{h} L\right)^2 = \frac{L^2}{h^2} x^2 \]

\[ \Rightarrow V = \int_{0}^{h} A(x) \, dx = \int_{0}^{h} \frac{L^2}{h^2} x^2 \, dx \]

\[ = \frac{L^2}{h^2} \cdot \frac{1}{3} h^3 \]

\[ = \frac{L^2 h}{3} \]
[Ex9] A wedge is cut out of a circular cylinder of radius 4 by two planes.

One plane is perpendicular to the axis of the cylinder. The other intersects the first at an angle of 30° along a diameter of the cylinder.

Find the volume of the wedge.

[Sol]:

\[ |BC| = y \tan 30° = \frac{1}{\sqrt{3}} y \]

\[ A(x) = \frac{1}{2} y \cdot \frac{1}{\sqrt{3}} y = \frac{1}{2\sqrt{3}} y^2 = \frac{16 - x^2}{2\sqrt{3}} \]

\[ V = \int_{-4}^{4} A(x) \, dx \]

\[ = \int_{-4}^{4} \frac{16 - x^2}{2\sqrt{3}} \, dx \]

\[ = \frac{1}{\sqrt{3}} \left[ 16x - \frac{1}{3} x^3 \right]_{0}^{4} \]

\[ = \frac{128}{3\sqrt{3}} \]
[Ex2] Find the volume of the solid obtained by rotating about the \(x\)-axis the region under the curve \(y = \sqrt{x}\) from 0 to 1.

[Sol]:

\[
\therefore A(x) = \pi y^2 = \pi (\sqrt{x})^2 = \pi x
\]
\[
\therefore V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \pi \cdot \frac{1}{2} x^2 \bigg|_0^1 = \frac{\pi}{2}
\]

[Ex3] Find the volume of the solid obtained by rotating the region bounded by \(y = x^3\), \(y = 8\), and \(x = 0\) about the \(y\)-axis.

[Sol]:

\[
\therefore A(y) = \pi x^2 = \pi (\sqrt[3]{y})^2 = \pi \frac{y}{3}
\]
\[
\therefore V = \int_0^8 A(y)\,dy = \int_0^8 \pi \frac{y^2}{3}\,dy = \pi \cdot \frac{3}{5} y^\frac{5}{3} \bigg|_0^8 = \frac{96\pi}{5}
\]
**[Ex4]** The region $R$ enclosed by the curve $y = x$ and $y = x^2$ is rotated about the $x$-axis. Find the volume of the resulting solid.

**[Sol]:**

\[
\therefore A(x) = \pi (x)^2 - \pi (x^2)^2 = \pi x^2 - \pi x^4
\]

\[
\therefore V = \int_0^1 \pi x^2 - \pi x^4 \, dx
\]

\[
= \pi \left( \frac{1}{3} x^3 - \frac{1}{5} x^5 \right) \bigg|_0^1 = \frac{2\pi}{15}
\]

**[Ex5]** Find the volume of the solid obtained by rotating the region in Ex4 about the line $y = 2$.

**[Sol]:**

\[
\therefore A(x) = \pi (2 - x^2)^2 - \pi (2 - x)^2
\]

\[
\therefore V = \int_0^1 \pi (2 - x^2)^2 - \pi (2 - x)^2 \, dx
\]

\[
= \pi \int_0^1 x^4 - 5x^2 + 4x \, dx
\]

\[
= \pi \left( \frac{1}{5} x^5 - \frac{5}{3} x^3 + 2x^2 \right) \bigg|_0^1 = \frac{8\pi}{15}
\]
[Ex6] Find the volume of the solid obtained by rotating the region in Ex4 about the line $x = -1$.

[Sol]:

\[
\begin{align*}
\therefore A(y) &= \pi (1 + \sqrt{y})^2 - \pi (1 + y)^2 \\
\therefore V &= \int_0^1 \pi (1 + \sqrt{y})^2 - \pi (1 + y)^2 \, dy \\
&= \pi \int_0^1 2\sqrt{y} - y - y^2 \, dy \\
&= \pi \left( \frac{4}{3}y^{\frac{3}{2}} - \frac{1}{2}y^2 - \frac{1}{3}y^3 \right)
\bigg|_0^1 \\
&= \frac{\pi}{2}
\end{align*}
\]

The solid in Example 2~6 are called solid of revolution 旋轉體.
§ 6.3 Volume by Cylindrical Shells

圆柱殼法求體積

\[ V \approx \sum_{i=1}^{n} 2\pi x_i f(x_i) \Delta x \]

\[ V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi \bar{x}_i f(\bar{x}_i) \Delta x \]
The volume of the solid obtained by rotating about the $y$-axis the region under the curve $y = f(x)$ from $a$ to $b$, is

$$V = \lim_{n \to \infty} \sum_{i=1}^{n} 2\pi x_i^* f(x_i^*) \Delta x = \int_{a}^{b} 2\pi x f(x) dx \text{ where } 0 \leq a < b$$

**[Ex1]** Find the volume of the solid obtained by rotating about the $y$-axis the region between $y = 2x^2 - x^3$ and $y = 0$.

**[Sol]:**

$$V = \int_{0}^{2} 2\pi x (2x^2 - x^3) dx$$

$$= 2\pi \int_{0}^{2} 2x^3 - x^4 dx$$

$$= 2\pi \left( \frac{1}{2} x^4 - \frac{1}{5} x^5 \right) \bigg|_{0}^{2}$$

$$= 2\pi \left( \frac{16}{2} - \frac{32}{5} \right)$$

$$= 2\pi \left( \frac{16 \cdot 5}{10} - \frac{32}{5} \right)$$

$$= 2\pi \left( \frac{8}{5} \right)$$

$$= \frac{16}{5} \pi$$
[Ex2] Find the volume of the solid obtained by rotating about the $y$-axis the region between $y = x$ and $y = x^2$.

[Sol 1] By method of cylindrical shells:

$$ V = \int_0^1 2\pi x(x - x^2) \, dx $$

$$ = 2\pi \int_0^1 x^2 - x^3 \, dx $$

$$ = 2\pi \left( \frac{1}{3} x^3 - \frac{1}{4} x^4 \right) \bigg|_0^1 $$

$$ = \frac{\pi}{6} $$

[Sol 2] By washer method:

$$ \because A(y) = \pi (\sqrt{y})^2 - \pi (y)^2 $$

$$ \therefore V = \int_0^1 \pi (\sqrt{y})^2 - \pi (y)^2 \, dy $$

$$ = \pi \int_0^1 y - y^2 \, dy $$

$$ = \pi \left( \frac{1}{2} y^2 - \frac{1}{3} y^3 \right) \bigg|_0^1 $$

$$ = \frac{\pi}{6} $$
[Ex3] Find the volume of the solid obtained by rotating about the x-axis the region under the curve $y = \sqrt{x}$ from 0 to 1.

[Sol 1] By cylindrical shells method:

$$V = \int_{0}^{1} 2\pi y (1 - y^2) dy$$

$$= 2\pi \int_{0}^{1} y - y^3 dy$$

$$= 2\pi \left( \frac{1}{2} y^2 - \frac{1}{4} y^4 \right) \bigg|_{0}^{1}$$

$$= \frac{\pi}{2}$$

[Sol 2] By disk method 圆盘法:

$$A(x) = \pi (\sqrt{x})^2$$

$$V = \int_{0}^{1} A(x) dx = \int_{0}^{1} \pi (\sqrt{x})^2 dx = \int_{0}^{1} \pi x dx$$

$$= \pi \cdot \frac{1}{2} x^2 \bigg|_{0}^{1} = \frac{\pi}{2}$$
[Ex4] Find the volume of the solid obtained by rotating the region bounded by $y = x - x^2$ and $y = 0$ about the line $x = 2$.

[Sol]:

\[
\begin{align*}
\begin{cases}
y = x - x^2 \\
y = 0
\end{cases}
\Rightarrow x^2 - x = 0 \Rightarrow x(x - 1) = 0 \Rightarrow x = 0, x = 1
\end{align*}
\]

\[
V = \int_0^1 2\pi (2 - x)(x - x^2) \, dx
\]

\[
= 2\pi \int_0^1 x^3 - 3x^2 + 2x \, dx
\]

\[
= 2\pi \left( \frac{1}{4} x^4 - x^3 + x^2 \right) \bigg|_0^1
\]

\[
= \pi - \frac{3\pi}{4}
\]

\[
= \frac{\pi}{2}
\]
§ 6.4 Work

If an object moves along a straight line with position function $s(t)$, then the force $F$ on the object (in the same direction) is defined by Newton’s Second Law of Motion as

$$F = m \frac{d^2 s}{dt^2}$$

$m$: mass (kg)
$s$: displacement (m)
$t$: time (s)
$F$: force (N=kg·m/s²) or pound (lb)

(1) In the case of \textit{constant acceleration} 加速度是常數, the force $F$ is also \textit{constant} and the work done is defined to be

$$W = F \cdot d$$

If $F$ is measured in \textit{newtons} and $d$ in \textit{meters}, then the unit for $W$ is a \textit{newton-meter} 牛頓-公尺, which is called a \textit{joule} 焦耳(J).

If $F$ is measured in \textit{pounds} and $d$ in \textit{feet}, then the unit for $W$ is a \textit{foot-pound} 英呎-磅(ft-lb), which is about 1.36J.
[Ex1] (a) How much work is done in lifting 1.2-kg book off the floor to put it on a desk that is 0.7m high? Use that fact that the acceleration due to gravity is $g = 9.8 \, \text{m/s}^2$.

(b) How much work is done in lifting a 20-lb weight 6ft off the ground?

[Sol]:

(a) The force exerted is equal and opposite to that exerted by gravity, so

$$F = mg = (1.2)(9.8) = 11.76 \, \text{N}$$

Therefore, the work done is

$$W = Fd = (11.76)(0.7) = 8.232 \, \text{J}$$

(b) In the case of $F = 20 \, \text{lb}$ and $d = 6 \, \text{ft}$, the work done is

$$W = Fd = (20)(6) = 120 \, \text{ft-lb}$$
(2) If the force is variable, let’s suppose that the object moves along the $x$-axis in the positive direction from $x = a$ to $x = b$, and at each point $x$ between $a$ and $b$ a force $f(x)$ acts on the object, where $f$ is a continuous function. We divide the interval $[a, b]$ into $n$ subintervals with endpoints $x_0 = a, x_1, \ldots, x_n = b$ and equal width $\Delta x$. Therefore we can approximate the total work by

$$W \approx \sum_{i=1}^{n} f(x^*_i)\Delta x , \text{ where } x^*_i \in [x_{i-1}, x_i] ,$$

We define the work done in moving the object from $a$ to $b$ as

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(x^*_i)\Delta x = \int_{a}^{b} f(x)dx$$

[Ex2] When a particle 質點 is located a distance $x$ feet from the origin, a force of $x^2 + 2x$ pounds acts on it. How much work is done in moving it from $x = 1$ to $x = 3$?

[Sol]: Since $f(x) = x^2 + 2x$, the work done in moving it from $x = 1$ to $x = 3$ is

$$W = \int_{1}^{3} f(x)dx = \int_{1}^{3} x^2 + 2xdx = \left( \frac{1}{3}x^3 + x^2 \right) \bigg|_{1}^{3} = \frac{50}{3} \text{ ft-lb}$$
By Hook’s Law 虎克定律

\[ f(x) = kx \], where \( k \) is the spring constant 彈性係數

[Ex3] A force of 40N is required to hold a spring that has been stretched 林展 from its natural length of 10cm to a length of 15cm. How much work is done in stretching the spring from 15cm to 18cm?

[Sol]:

According to Hook’s Law, the force required to hold the spring stretched 林展 meters beyond its natural length is \( f(x) = kx \).

When the spring is stretched from 10cm to 15cm, \( x = \frac{15-10}{100} = 0.05 \) m

\[ f(0.05) = 40 \text{ i.e. } 0.05k = 40 \Rightarrow k = 800. \]

Thus, \( f(x) = 800x \) and the work done in stretching the spring from 15cm to 18cm is

\[
W = \int_{0.05}^{0.08} 800x\,dx = 400x^2 \bigg|_{0.05}^{0.08} = 400\left[(0.08)^2 - (0.05)^2\right] = 1.56 \text{ J}
\]
[Ex4] A 200-lb cable is 100ft long and hangs 垂掛 vertically 垂直 from the top of a tall building. How much work is required to lift 拉 the cable to top of the building?

[Sol]:

We don’t have a formula for the force function here, so let’s use the initial approach.

First, we divide the cable into small parts of equal length $\Delta x = \frac{100}{n}$. So the weight of the $i^{th}$ part is $2\Delta x$ lb since the cable weights $\frac{200}{100} = 2$ pounds per foot. Thus, the work done on the $i^{th}$ part is

$$(2\Delta x) \cdot x_i^* = 2x_i^*\Delta x$$

So, $W \approx \sum_{i=1}^{n} 2x_i^*\Delta x$

$$\Rightarrow W = \lim_{n \to \infty} \sum_{i=1}^{n} 2x_i^*\Delta x = \int_{0}^{100} 2xdx$$

$$= x^2 \bigg|_{0}^{100} = 10000 \text{ ft-lb}$$
A tank has the shape of an inverted circular cone 倒圆锥形水槽 with height 10m and base radius 4m. It is filled with water to a height of 8m. Find the work required to empty the tank by pumping 抽 all of the water to the top of the tank. (The density 密度 of water is 1000 kg/m³)

[Sol]:

We divide the interval [2,10] into n subintervals of equal width Δx with endpoints $x_0 = 2, x_1, \ldots, x_n = 10$ and choose $x_i^*$ in the $i$th subinterval. The $i$th layer is approximated by a circular cylinder with radius $r_i$ and height $Δx$.

$$\frac{r_i}{10 - x_i^*} = \frac{4}{10} \implies r_i = \frac{2}{5} (10 - x_i^*)$$

The volume of the $i$th layer

$$V_i \approx \pi r_i^2 Δx = \frac{4\pi}{25} (10 - x_i^*)^2 Δx$$
So its mass $m_i \approx 1000 \frac{4\pi}{25} (10 - x_i^*)^2 \Delta x = 160\pi (10 - x_i^*)^2 \Delta x$

The force required to raise the $i$th layer is

$$F_i = m_i g \approx 160\pi (10 - x_i^*)^2 \Delta x \cdot (9.8) = 1568\pi (10 - x_i^*)^2 \Delta x$$

The work done to raise the $i$th layer to the top is

$$W_i \approx F_i x_i^* \approx 1568\pi x_i^* (10 - x_i^*)^2 \Delta x$$

Therefore, the total work done is

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} W_i$$

$$= \lim_{n \to \infty} \sum_{i=1}^{n} 1568\pi x_i^* (10 - x_i^*)^2 \Delta x$$

$$= \int_{2}^{10} 1568\pi x (10 - x)^2 \, dx$$

$$= \cdots$$

$$= 1568\pi \cdot \frac{2048}{3} \approx 3.36 \times 10^6 \text{ J}$$
The average value of \( f \) on the interval \([a, b]\) is defined as
\[
f_{\text{ave}} = \frac{1}{b-a} \int_{a}^{b} f(x) \, dx
\]

[Ex1] Find the average value of the function \( f(x) = 1 + x^2 \) on the interval \([-1, 2]\).

[Sol]:
\[
f_{\text{ave}} = \frac{1}{2-(-1)} \int_{-1}^{2} (1 + x^2) \, dx = \frac{1}{3} \left( x + \frac{1}{3} x^3 \right) \bigg|_{-1}^{2} = 2
\]

The Mean Value Theorem for Integrals
If \( f \) is continuous on \([a, b]\), then there exists a number \( c \) in \([a, b]\) such that
\[
\int_{a}^{b} f(x) \, dx = f(c)(b-a)
\]
or \( f_{\text{ave}} = f(c) \)
[Ex2] \( f(x) = 1 + x^2 \) is continuous on \([-1, 2]\). Find all number \( c \) that satisfy the conclusion of the Mean Value Theorem for Integrals.

[Sol]:

According to the Mean Value Theorem for Integrals, there is a number \( c \) in \([-1, 2]\) such that.

\[
\int_{-1}^{2} (1 + x^2) \, dx = f(c)(2 - (-1)) \Rightarrow 6 = f(c) \cdot 3
\]

\Rightarrow f(c) = 2 = f_{\text{ave}}

i.e. \( 1 + c^2 = 2 \Rightarrow c = \pm 1 \in [-1, 2] \)

There happen to be two numbers \( c = 1 \) and \( c = -1 \) in \([-1, 2]\) that work in the Theorem.
[Ex3] Show that the average velocity 速度 of a car over a time interval \([t_1, t_2]\) is the same as the average of its velocities during the trip.

[Sol]: If \(s(t)\) is the displacement of the car at time \(t\), then the average velocity of the car over \([t_1, t_2]\) is

\[
\frac{\Delta s}{\Delta t} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}
\]

On the other hand, the average value of the velocity function on \([t_1, t_2]\) is

\[
\nu_{\text{ave}} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} v(t) dt = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} s'(t) dt
\]

\[
= \frac{1}{t_2 - t_1} \left[ s(t_2) - s(t_1) \right] = \text{average velocity}
\]

(by the Net Change Theorem) 淨值定理