Microwave Engineering

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Outline

1. Transmission Line Theory
2. Transmission Lines and Waveguides
   - General Solutions for TEM, TE, and TM waves; Parallel Plate waveguide; Rectangular Waveguide; Coaxial Line; Stripline; Microstrip
3. Microwave Network Analysis
   - Impedance and Equivalent Voltages and Currents; Impedance and Admittance Matrices; The Scattering Matrix; ABCD Matrix; Signal Flow Graphs; Discontinuities and Model Analysis
4. Impedance Matching and Tuning
   - Matching with Lumped Elements; Single-Stub Tuning; Double-Stub Tuning; The Quarter-Wave Transformer; The Theory of Small Reflections
5. Microwave Resonators
   - Series and Parallel Resonant Circuits; Transmission Line Resonators; Rectangular Waveguide Cavities; Dielectric Resonators
6. Power Dividers and Directional Couplers
   - Basic Properties of Dividers and Couplers; The T-Junction Power Divider; The Wilkinson Power Divider; Coupled Line Directional Couplers; 180° hybrid
7. Microwave Filters
   - Periodic Structure; Filter Design by the Insertion Loss Method; Filter Transformations; Filter Implementation;
4. Impedance Matching and Tuning

Matching with Lumped Elements
Single-Stub Tuning
Double-Stub Tuning
The Quarter-Wave Transformer
The Theory of Small Reflections
Tapered Lines
Impedance matching is often a part of the larger design process for a microwave component or system, and the matching circuit is ideally lossless, to avoid unnecessary loss of power and is usually designed so that the impedance seen looking into the matching network is $Z_0$. Then reflections are eliminated on the transmission line to the left of the matching network.

**Impedance matching or tuning is important for the following reasons:**

(I) Maximum power is delivered when the load is matched to the line (assuming the generator is matched), and power loss in the feed line is minimized.

(II) Impedance matching sensitive receiver components (antenna, low-noise amplifier, etc) improves the signal-to-noise ratio of the system.

(III) Impedance matching in a power distribution network (such as an antenna array feed network) will reduce amplitude and phase errors.

As long as the load impedance, $Z_L$, has some nonzero real part. Factors that may be important in the section of a particular matching network include the following:

**Complexity** - the simplest design that satisfies the required specifications is generally the most preferable.

**Bandwidth**

**Implementation** - depending on the type of transmission line or waveguide being used; tuning stubs are much easier to implement in waveguide.

**Adjustability** – adjustment to match a variable load impedance

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A lossless network matching an arbitrary load impedance to a transmission line.
Matching with Lumped Elements (L Networks)

Simplest type of matching network is the L-section, which uses two reactive elements to match an arbitrary load impedance to a transmission line.

=> If the normalized load impedance, \( z_L = \frac{Z_L}{Z_o} \), is inside the \( 1 + jx \) circle on the Smith Chart, then the circuit as the fig. (a).

⇒If the normalized load impedance is outside the \( 1 + jx \) circle on the Smith Chart, then the circuit as the fig. (b).

The \( 1 + jx \) circle is the resistance circle on the impedance Smith Chart for which \( r = 1 \). In addition, the reactive elements may be either inductors or capacitors, depending on the load impedance. (eight distinct possibilities for the matching circuit for various load impedance)
• Smith chart solution

Z-plane
CW → series L
CCW → series C

Y-plane
CW → shunt C
CCW → shunt L
Analytic Solutions

Consider the circuit of Fig. (a), let $Z_L = R_L + j X_L$ => We stated that this circuit would be used when $z_L = Z_L / Z_0$ is inside the $1 + j x$ circle on the Smith chart => implies that $R_L > Z_0$

$\Rightarrow$ The impedance seen looking into the matching network followed by the load impedance must be equal to $Z_0$, for a match

$$Z_0 = jX + \frac{1}{jB + 1/(R_L + jX_L)}$$

Rearranging and separating into real and imaginary parts gives two equations for the two unknowns, $X$ and $B$

$$B(XR_L - X_L Z_0) = R_L - Z_0 \quad ; \quad X(1 - BX_L) = BZ_0 R_L - X_L$$

Solving above equation for $X$ and substituting into above equation gives a quadratic equation for $B$.

The solution is

$$B = \frac{X_L \pm \sqrt{R_L / Z_0 \sqrt{R_L^2 + X_L^2 - Z_0 R_L}}}{R_L^2 + X_L^2}$$

Since $R_L > Z_0$, the argument of the second square root is always positive.

Then the series reactance can be found as

$$X = \frac{1}{B} + \frac{X_L Z_0}{R_L} - \frac{Z_0}{BR_L}$$

Two solutions are possible for $B$ and $X$. Both of these solutions are physically realizable, since both positive and negative values of $B$ and $X$ are possible. (positive $X$ implies an inductor, negative $X$ implies a capacitor, while positive $B$ implies a capacitor and negative $B$ implies an inductor)
**ZL** inside 1+jx circle, two possible solutions

Smith chart solution

1+jx circle

analytical solution

\[ Z_o = jX + \frac{1}{jB + \frac{1}{R_L + jX_L}} \]

\( \Rightarrow B > 0 \rightarrow C, B < 0 \rightarrow L \)

\( X > 0 \rightarrow L, X < 0 \rightarrow C \)
Consider the circuit of Fig. (b), let \( Z_L = R_L + jX_L \) => We stated that this circuit would be used when \( z_L = Z_L / Z_0 \) is outside the 1 + j\( x \) circle on the Smith chart => implies that \( R_L < Z_0 \)

\( \Rightarrow \) The admittance seen looking into the matching network followed by the load impedance must be equal to \( 1 / Z_0 \), for a match

\[
\frac{1}{Z_0} = jB + \frac{1}{R_L + j(X + jX_L)}
\]

Rearranging and separating into real and imaginary parts gives two equations for the two unknowns, \( X \) and \( B \)

\[
BZ_0(X + X_L) = Z_0 - R_L ; \quad (X + X_L) = BZ_0 R_L
\]

Solving above equation for \( X \) and substituting into above equation gives a quadratic equation for \( B \).

The solution is

\[
B = \frac{X_L \pm \sqrt{R_L / Z_0 \sqrt{R_L^2 + X_L^2 - Z_0 R_L}}}{R_L^2 + X_L^2} ; \quad B = \pm \frac{\sqrt{(Z_0 - R_L) / R_L}}{Z_0}
\]

Since \( R_L < Z_0 \), the argument of the square roots are always positive.

In order to match an arbitrary complex load to a line of characteristic impedance \( Z_0 \), the real part of the input impedance to the matching network must be \( Z_0 \) while the imaginary part must be zero. This implies that a general matching network must have at least two degrees of freedom => in the L-section matching circuit these two degrees of freedom are provided by the values of two reactive components.

\[ \text{L-section matching networks. (a) Network for } z_L \text{ inside the 1 + j} x \text{ circle.} \]

\[ \text{(b) Network for } z_L \text{ outside the 1 + j} x \text{ circle.} \]
$Z_L$ outside $1+jx$ circle, two possible solutions

Smith chart solution

analytical solution

$$\frac{I}{Z_0} = jB + \frac{1}{R_L + j(X + X_L)}$$

$B > 0 \rightarrow C, B < 0 \rightarrow L$

$X > 0 \rightarrow L, X < 0 \rightarrow C$
EX: Design an L-section matching network to match a series RC load with an impedance $Z_L = 200 - j 100 \, \Omega$, to a 100 $\Omega$ line, at a frequency of 500MHz.

1. $z_1 = 2 - j 1$, $y_1 = 0.4 + j 0.2$
   Solution A

2. $y = 0.4 + j 0.5 \rightarrow jb = j 0.3 \rightarrow jB = j \omega C = jb / Z_0$
   
   \[
   C = \frac{b}{Z_0 \omega} = 0.292 \text{pF}
   \]

   $z = 1 - j 1.2 \rightarrow jx = j 1.2 \rightarrow jX = j \omega L = j x Z_0$

   \[
   L = x Z_0 / \omega = 38.8 \text{nH}
   \]
   Solution B

3. $y = 0.4 - j 0.5 \rightarrow jb = - j 0.7 \rightarrow jB = 1 / j \omega L = -jb / Z_0$
   
   \[
   L = - Z_0 / \omega b = 46.1 \text{nH}
   \]

   $z = 1 + j 1.2 \rightarrow jx = -j 1.2 \rightarrow jX = 1 / j \omega C = -j x Z_0$

   \[
   C = - 1 / x Z_0 \omega \approx 2.61 \text{pF}
   \]
Single-Stub Tuning

It is another consider a matching technique that uses a single open-circuited or short-circuited length of a transmission line (a “stub”), connected either in parallel or in series with the transmission feed line at a certain distance from the load.

⇒ The two adjustable parameters are the distance, d, from the load to the stub position, and the value of susceptance or reactance provided by the shunt or series stub. [ For shunt : to select d so that the admittance, Y, seen looking into the line at distance d from the load is of the form $Y_0 + jB$ - then the stub susceptance is chosen as $-jB$ ] ; [ For series : the distance d is selected so that the impedance, Z, seen looking into the line at a distance d from the load, is of the form $Z_0 + jX$ -> then the stub reactance is chosen as $-jX$ ]

The proper length of open or shorted TL can be provide any desired value of reactance and impedance

Can match various impedance (real part ≠ 0)
• equivalent microstrip elements
  - series C
  - series L  series a high impedance microstrip line
  - shunt C  shunt an open microstrip line
  - shunt L  shunt a short microstrip line

An open microstrip line

\[ Z_{in} = \frac{Z_0}{j \tan \beta l} \equiv \frac{l}{j \omega C} \]

A short microstrip line

\[ Z_{in} = jZ_0 \tan \beta l \equiv j \omega L \]

A high impedance microstrip line
Shunt stubs

Shunt stub

\[ y_{in} = j \beta = j \omega C \]

\[ y_{in} = l - j \beta \]

\[ y_{in} = l \]

\[ y_{in} = 1 + j \beta \]

Smith chart solution

constant \( \Gamma \)-circle
EX. For a load impedance $Z_L = 60-j80 \ \Omega$, design two single-stub (short circuit) shunt tuning networks to match this load to a 50 $\Omega$ line. Assuming that the load is matched at 2GHz, and the load consists of a resister and capacitor in series, plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

$<$Sol$>$ normalized load impedance: $z_L = 1.2-j1.6$ -> construct the appropriate SWR circle, and convert to the load admittance, $y_L$. -> Now notice that SWR circle intersects the $1+jb$ circle at two points ($y_1$ and $y_2$). Thus the distance $d$, from the load to the stub, is given by either of these two intersections. Reading the WTG scale $\Rightarrow d_1 = 0.176 - 0.065 = 0.110\lambda$ ; $d_2 = 0.325 - 0.065 = 0.260\lambda$

$\Rightarrow$ There is an infinite number of distances, $d$, on the SWR circle that intersect the $1+jb$ circle. $\Rightarrow$ Usually, it is desired to keep the matching stub as close as possible to the load, to improve the bandwidth of the match and reduce losses caused by a possibly large standing wave ratio on the line between the stub and the load.

$\Rightarrow y_1 = 1.00 + j1.47$ ; $y_2 = 1.00 - j1.47$ $\Rightarrow$ susceptance of $-j1.47$ and $j1.47$ $\Rightarrow l_1 = 0.095\lambda$ ; $l_2 = 0.405\lambda$

$\Rightarrow$ load impedance $\Rightarrow R = 60\Omega$ and $C = 0.995 \mu F$
Solution to Example 5.2.
(a) Smith chart for the shunt-stub tuners.
To derive formulas for \( d \) and \( l \), let the load impedance be written as \( Z_L = 1/Y_L = R_L + jX_L \)

\[ \Rightarrow \text{the impedance } Z \text{ down a length, } d, \text{ of line from the load is} \]

\[ Z = Z_o \left( \frac{R_L + jX_L}{Z_o + j(R_L + jX_L)t} \right) \text{ where } t = \tan \beta d \]

The admittance at this point is \( Y = G + jB = 1/Z \), where

\[ G = \frac{R_L(1+t^2)}{R_L^2 + (X_L + Z_o t)^2}, B = \frac{R_L^2 t - (Z_o - X_L t)(X_L + Z_o t)}{Z_o[R_L^2 + (X_L + Z_o t)^2]} \]

\[ \Rightarrow \text{Now } d \text{ (which implies } t) \text{ is chosen so that } G = Y_o = 1/Z \]

\[ \Rightarrow Z_o(R_L - Z_o)t^2 - 2X_LZ_o t + (R_LZ_o - R_L^2 - X_L^2) = 0 \quad \Rightarrow \quad t = \frac{X_L \pm \sqrt{R_L\left((Z_o - R_L)^2 + X_L^2\right)}}{R_L - Z_o}, \text{ for } R_L \neq Z_o \]

If \( R_L = Z_o \), then \( t = -X_L/2Z_o \)

\[ \Rightarrow \frac{d}{\lambda} = \begin{cases} \frac{1}{2\pi} \tan^{-1} t, & \text{for } t \geq 0 \\ \frac{1}{2\pi} (\pi + \tan^{-1} t), & \text{for } t < 0 \end{cases} \]

\[ \Rightarrow \begin{cases} \text{for an open-circuited stub} : \frac{l_o}{\lambda} = \frac{1}{2\pi} \tan^{-1} \left( \frac{B_s}{Y_o} \right) = -\frac{1}{2\pi} \tan^{-1} \left( \frac{B}{Y_o} \right) \\ \text{for a short-circuited stub} : \frac{l_s}{\lambda} = -\frac{1}{2\pi} \tan^{-1} \left( \frac{Y_o}{B_s} \right) = \frac{1}{2\pi} \tan^{-1} \left( \frac{Y_o}{B} \right) \end{cases} \]

where \( B_s = -B \)
Series Stubs

Series stub

\[
Z_{in} = 1 + jx
\]

\[
Z_o \quad Z_o \quad Z_L
\]

\[
z_{in} = -jx = \frac{1}{j\omega C}
\]

\[
l
\]

\[
Z_o \quad Z_o \quad Z_L
\]

\[
z_{in} = 1 - jx
\]

\[
l
\]

Smith chart solution

\[
G
\]

\[
G
\]

\[
z_{in} = jx = j\omega L
\]
EX. Match a load impedance of $Z_L=100+j80$ $\Omega$ to a 50$\Omega$ line using a single series open-circuit stub. Assuming that the load is matched at 2GHz, and the load consists of a resistor and inductor in series, plot the reflection coefficient magnitude from 1 GHz to 3 GHz for each solution.

$<$Sol$>$ normalized load impedance: $z_L=2-j1.6$ -> draw the SWR circle. For the series-stub design, the chart is an impedance chart. -> Now notice that SWR circle intersects the 1+jx circle at two points ($z_1$ and $z_2$). The shortest distance, $d_1$, from the load to the stub is, from the WTG scale, $=>$ $d_1 = 0.328 - 0.208 = 0.120\lambda$; while the second distance is $d_2 = (0.5 - 0.208) + 0.172 = 0.463\lambda$.

$\Rightarrow$ $z_1 = 1.00 - j1.33$ ; $z_2 = 1.00 + j1.33$ $\Rightarrow$ reactance of $j1.33$ and $-j1.33$ $\Rightarrow$ $l_1 = 0.397\lambda$ ; $l_2 = 0.103\lambda$

$\rightarrow$ load impedance $=>$ $R = 100\Omega$ and $L = 6.37$ pH
Solution to Example 5.3.
(a) Smith chart for the series-stub tuners.
To derive formulas for $d$ and $l$, for the series - stub tuner, let the load admittance be written as $Y_L = 1/Z_L = G_L + jB_L$

$=>$ the admittance $Y$ down a length, $d$, of line from the load is

$=> Y = Y_o \left( \frac{G_L + jB_L}{Y_o + j(G_L + jB_L)} \right)^{d}$ where $t = \tan \beta d$, and $Y_o = 1/Z_o$

The admittance at this point is $Z = R + jX = 1/Y$, where $R = \frac{G_L(1+t^2)}{G_L^2 + (B_L + Y_o t)^2}$; $X = \frac{G_L^2 t - (Y_o - B_L t)(B_L + Y_o t)}{Y_o [G_L^2 + (B_L + Y_o t)^2]}$

$=>$ Now $d$ (which implies $t$) is chosen so that $R = Z_o = 1/Y$

$=> Y_o (G_L - Y_o)^2 - 2B_L Y_o t + (G_L Y_o - G_L^2 - B_L^2) = 0$  $=>$  $t = \frac{B_L \pm \sqrt{G_L (Y_o - G_L)^2 + B_L^2}}{G_L - Y_o}$, for $G_L \neq Y_o$

If $G_L = Y_o$, then $t = -B_L/2Y_o$  $=>$  $\frac{d}{\lambda} = \left\{ \begin{array}{ll}
\frac{1}{2\pi} \tan^{-1} \frac{t}{\pi + \tan^{-1} t}, & for \ t \geq 0 \\
\frac{1}{2\pi} \tan^{-1} \frac{t}{\pi + \tan^{-1} t}, & for \ t < 0
\end{array} \right.$

$=>$  $\frac{l_o}{\lambda} = \frac{1}{2\pi} \tan^{-1} \left( \frac{X_l}{Z_o} \right) = -\frac{1}{2\pi} \tan^{-1} \left( \frac{X}{Z_o} \right)$

for a short - circuited stub

$=>$  $\frac{l_s}{\lambda} = \frac{-1}{2\pi} \tan^{-1} \left( \frac{Z_o}{X_s} \right) = \frac{1}{2\pi} \tan^{-1} \left( \frac{Z_o}{X} \right)$  where $X_s = -X$

for an open - circuited stub
Double-Stub Tuning

The double stub tuner, which uses two tuning stubs in fixed positions, can be used. Such tuners are often fabricated in coaxial line, with adjustable stubs connected in parallel to the main coaxial line. -> the double stub tuner cannot match all load impedances.

The double stub tuner circuit, where the load may be an arbitrary distance from the first stub. Although this is more representative of practical situation, where the load $Y_L'$ has been transformed back to the position of the first stub, is easier to deal with and does not lose any generality. => the stubs are shunt stubs, which are usually easier to implement in practice than are series stubs. (open-circuited or short-circuited)

$\Rightarrow d \rightarrow \lambda /8$ or $3\lambda /8$
Smith Chart Solution

1. The susceptance of the first stub, $b_1 (b_1')$, moves the load admittance to $y_1 (y_1')$ => These points lie on the rotated $1+ jb$ circle; the amount of rotation is $d$ wavelength toward the load, where $d$ is the electrical distance between the two stub.

2. Transforming $y_1 (y_1')$ toward the generator through a length, $d$, of line leaves us at the point $y_2 (y_2')$ which must be on the $1+ jb$ circle. => the second stub then adds a susceptance $b_2 (b_2')$, which brings us to the center of the chart, and complete match.
EX. Design a double stub shunt tuner to match a load impedance $Z_L = 60 - j80 \, \Omega$ to a 50 $\Omega$ line. The stubs are to be open-circuited stubs, and are spaced $\lambda / 8$ apart. Assuming that this load consists of a series resistor and capacitor, and that the match frequency is 2 GHz, plot the reflection coefficient magnitude versus frequency from 1 GHz to 3 GHz.

$$ \text{<Sol> normalized load admittance: } y_L = 0.3 + j0.4. \quad \Rightarrow \text{ construct the rotated } 1 + jb \text{ conductance circle, by moving every point on the } g = 1 \text{ circle } \lambda / 8 \text{ toward the load} \Rightarrow \text{ the first stub: } b_1 = 1.314 \Rightarrow b_1' = -0.114$$

$\Rightarrow$ Transform the $\lambda / 8$ section of line by rotating along a constant radius (SWR) circle $\lambda / 8$ toward the generator $\Rightarrow y_2 = 1 - j3.38 \Rightarrow y_2' = 1 + j1.38$

$\Rightarrow$ The susceptance of the second stub should be $b_2 = 3.38 \Rightarrow b_2' = -1.38$

$\Rightarrow$ The lengths of the open-circuited stubs are then found as $l_1 = 0.146\lambda \Rightarrow l_1' = 0.204\lambda ; l_2 = 0.482\lambda \Rightarrow l_2' = 0.350\lambda$

$\Rightarrow$ load impedance $\Rightarrow R = 60\Omega$ and $C = 0.995 \, \text{pF}$
Solution to Example 5.4.
(a) Smith chart for the double-stub tuners.
The left of the first stub, the admittance is \( Y_1 = G_L + j(B_L + B_1) \) where \( Y_L = G_L + jB_L \) is the load admittance and \( B_1 \) is the susceptance of the first stub.

After transforming through a length \( d \) of transmission line, the admittance just to the right of the second stub is

\[
\Rightarrow Y_2 = Y_o \frac{G_L + j(B_L + B_1 + Y_o t)}{Y_o + j(t)(G_L + jB_L + jB_1)} \text{ where } t = \tan \beta d \text{, and } Y_o = 1 / Z_o
\]

At this point, the real part of \( Y_2 \) must equal \( Y_o \), which leads to the equation

\[
\Rightarrow G_L^2 - G_L Y_o \frac{1 + t^2}{t^2} + \frac{(Y_o - B_L t - B_1 t)^2}{t^2} = 0 \Rightarrow \text{Solving for } G_L : G_L = Y_o \frac{1 + t^2}{2t^2} \left[ 1 \pm \frac{4t^2(Y_o - B_L t - B_1 t)}{Y_o^2(1 + t^2)^2} \right]
\]

Since \( G_L \) is real, the quantity within the square root must be nonnegative

\[
\Rightarrow 0 \leq \frac{4t^2(Y_o - B_L t - B_1 t)^2}{Y_o^2(1 + t^2)^2} \leq 1 \Rightarrow 0 \leq G_L \leq Y_o \frac{1 + t^2}{t^2} = \frac{Y_o}{\sin^2 \beta d}
\]

which gives the range on \( G_L \) that can be matched for a given stub spacing, \( d \).

After \( d \) has been fixed, the first stub susceptance can be determined

\[
\Rightarrow B_1 = -B_L + Y_o \frac{Y_o \pm \sqrt{(1 + t^2)G_L Y_o - G_L^2 t^2}}{t}
\]

the second stub susceptance \( \Rightarrow B_2 = \frac{\pm Y_o \sqrt{Y_o G_L (1 + t^2) - G_L^2 t^2} + G_L Y_o}{G_L t} \)

\[
\begin{align*}
\text{for open - circuited stub} : & \quad l_o = \frac{1}{2\pi} \tan^{-1} \left( \frac{B}{Y_o} \right) \\
\text{for short - circuited stub} : & \quad l_o = \frac{-1}{2\pi} \tan^{-1} \left( \frac{Y_o}{B} \right) \\
\text{where } B = B_1 \text{ or } B_2
\end{align*}
\]
The quarter-wave transformer

The quarter-wave transformer is a simple and useful circuit for matching a real load impedance to a transmission line. An additional feature of the quarter-wave transformer is that it can be extended to multisection designs in a methodical manner, for broader bandwidth. => If only a narrow band impedance match is required, a single-section transformer may suffice.

One drawback of the quarter-wave transformer is that it can only match a real load impedance. => A complex load impedance can always be transformed to a real impedance by using an appropriate length of transmission line between the load and the transformer, or an appropriate series or shunt reactive stubs.

\[ Z_L = \sqrt{Z_0 Z_T} \]

\[ |f'(\theta)| = \frac{|Z_L - Z_0|}{2\sqrt{Z_0 Z_L}} \cos \theta \] for \( \theta \) near \( \theta_0 = \frac{\pi}{2} \), \( \theta = \beta l \)

\[ \frac{\Delta f}{f_0} = 2 \frac{\cos^{-1} \left( \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \right)}{\pi} \frac{2 \sqrt{Z_0 Z_L}}{|Z_L - Z_0|}, Z_L \to Z_0, \Delta f \text{ increases} \]
The single section quarter wave matching transformer circuit is shown in Figure.

The characteristic impedance of the matching section is
\[ Z_i = \sqrt{Z_o Z_L} \]

\[ \Rightarrow \text{At the design frequency, } f_o, \text{ the electrical length of the matching section is } \lambda_o / 4, \]

but at other frequencies the length is different

\[ \Rightarrow \text{so a prefect match is no longer obtained} \]

\[ \Rightarrow \text{approximate expression for the mismatch versus frequency} \]

The input impedance seen looking into the matching section is
\[ Z_i = Z_i \frac{Z_L + j Z_L t}{Z_i + j Z_L t} \]

where \( t = \tan \beta l = \tan \theta, \) and \( \beta l = \theta = \pi / 2 \) at the design frequency, \( f_o. \)

\[ \Rightarrow \Gamma = \frac{Z_i - Z_o}{Z_i + Z_o} = \frac{\left( \frac{Z_L}{Z_i} - Z_o \right) + j \left( \frac{Z_L^2}{Z_i} - Z_o Z_L \right)}{\left( \frac{Z_L}{Z_i} + Z_o \right) + j \left( \frac{Z_L^2}{Z_i} + Z_o Z_L \right)} \]

Since \( Z_i^2 = Z_o Z_L, \) this reduces to
\[ \Gamma = \frac{Z_L - Z_o}{Z_i + Z_o + j 2 t \sqrt{Z_o Z_L}} \]

\[ \Rightarrow \text{the reflection coefficient magnitude is} \]
\[ |\Gamma| = \frac{|Z_L - Z_o|}{\left( \left( \frac{Z_L}{Z_i} + Z_o \right)^2 + 4 t^2 Z_o Z_L \right)^{1/2}} = \frac{1}{\left[ 1 + 4 Z_o Z_L (Z_L - Z_o)^2 \right]^{1/2}} \]

Now if we assume that the frequency is near the design frequency, \( f_o, \) then \( l \approx \lambda_o / 4 \) and \( \theta \approx \pi / 2. \) Then \( \sec^2 \theta \gg 1 \)

\[ \Rightarrow |\Gamma| \approx \frac{|Z_L - Z_o| \cos \theta}{2 \sqrt{Z_L Z_o}}, \text{ for } \theta \text{ near } \pi / 2 \]

If we assume TEM lines, then \( \theta = \beta l = \frac{2 \pi f}{v_p} \frac{v_p}{4 f_o} = \frac{\pi f}{2 f_o}, \) therefore the frequency of the lower band edge at \( \theta = \theta_m \)

\[ \Rightarrow \text{and the fractional band width is} \]
\[ \Delta f = \frac{2 (f_o - f_m)}{f_o} = 2 - \frac{2 f_m}{f_o} = 2 - \frac{4 \theta_m}{\pi} = 2 - \frac{4 \cos^{-1} \left[ \frac{\Gamma_m}{\sqrt{1 - \Gamma_m^2}} \frac{2 \sqrt{Z_o Z_L}}{|Z_L - Z_o|} \right]}{\pi} \]
The theory of small reflection

Single-section Transformer

\[ \Gamma_1 = \frac{(Z_2 - Z_1)}{(Z_2 + Z_1)} \quad \Gamma_2 = -\Gamma_1 \quad \Gamma_3 = \frac{(Z_L - Z_2)}{(Z_L + Z_2)} \]

\[ T_{21} = 1 + \Gamma_1 = 2Z_2/(Z_1 + Z_2) \quad T_{12} = 1 + \Gamma_2 = 2Z_1/(Z_1 + Z_2) \]

\[ \Gamma_2 = -\Gamma_1 \quad T_{21} = I + \Gamma_1 \quad T_{12} = I + \Gamma_2 \]

\[ \Gamma_m = \frac{T_{12}T_{21}\Gamma_3 e^{-j2\theta}}{1 - \Gamma_2 \Gamma_3 e^{-j2\theta}} = \frac{\Gamma_1 + \Gamma_3 e^{-j2\theta}}{1 + \Gamma_1\Gamma_3 e^{-j2\theta}} \]

\[ \approx \Gamma_1 + \Gamma_3 e^{-j2\theta} \text{ if } Z_1 \approx Z_2 \approx Z_L \]
Partial reflection coefficients can be defined at each junctions

\[ \Gamma_o = \frac{(Z_1 - Z_0)(Z_1 + Z_0)}{(Z_1 + Z_0)} \; ; \; \Gamma_n = \frac{(Z_{n+1} - Z_n)(Z_{n+1} + Z_n)}{(Z_{n+1} + Z_n)} \; ; \]
\[ \Gamma_N = \frac{(Z_L - Z_N)(Z_L + Z_N)}{(Z_L + Z_N)} \; ; \]

\[ \Gamma(\theta) = \Gamma_0 + \Gamma_1 e^{-j2\theta} + \Gamma_2 e^{-j4\theta} + \ldots + \Gamma_N e^{-j2N\theta} \]

\[
\begin{cases} 
2 e^{-jN\theta} \left[ \Gamma_o \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots + \Gamma_n \cos(N-2n)\theta + \ldots + \frac{1}{2} \Gamma_{(N-1)/2} \cos \theta \right] & \text{N odd} \\
2 e^{-jN\theta} \left[ \Gamma_o \cos N\theta + \Gamma_1 \cos(N-2)\theta + \ldots + \Gamma_n \cos(N-2n)\theta + \ldots + \frac{1}{2} \Gamma_{N/2} \right] & \text{N even} 
\end{cases}
\]
Taped Lines

Discuss how an arbitrary real load impedance could be matched to a line over a desired bandwidth by using multisection matching transformers. As the number, $N$, of discrete sections increases, the step changes in characteristic impedance between the sections become smaller. Thus, in the limit of an infinite number of sections, we approach a continuously tapered line.

Frequency response

\[
\Delta \Gamma = \frac{Z + \Delta Z - Z}{Z + \Delta Z + Z} \approx \frac{\Delta Z}{2Z}
\]

\[
\Rightarrow d\Gamma = \frac{1}{2} \frac{dZ}{Z} = \frac{d \ln \frac{Z}{Z_o}}{dz}dz
\]

\[
\Rightarrow \Gamma(\theta) = \frac{1}{2} \int_0^\theta e^{-j\beta z} \frac{d}{dz} \left( \ln \frac{Z}{Z_o} \right) dz
\]

A tapered transmission line matching section and the model for an incremental length of tapered line.

(a) The tapered transmission line matching section.

(b) Model for an incremental step change in impedance of the tapered line.
Exponential taper

\[ Z(z) = Z_o e^{az} \quad 0 < z < L \]

\[ Z(L) = Z_L = Z_o e^{al} \quad \rightarrow a = \frac{1}{L} \ln \frac{Z_L}{Z_o} \]

\[ \Gamma(\theta) = \frac{1}{2} \left[ e^{-j2\beta L} \frac{d}{dz} (\ln e^{az})dz \right] = \frac{1}{2} \ln \frac{Z_L}{Z_o} e^{-j\beta L} \frac{\sin \beta L}{\beta L} \]

\[ L \uparrow, \Gamma(\theta) \downarrow \]

Triangular taper

\[ Z(z) = \begin{cases} 
Z_o e^{2(z/L)^2 \ln Z_L/Z_o} & 0 < z < L/2 \\
Z_o e^{(4z/L - 2z^2/L^2 - 1)^2 \ln Z_L/Z_o} & L/2 < z < L
\end{cases} \]

\[ \Gamma(\theta) = \frac{1}{2} \ln \frac{Z_L}{Z_o} e^{-j\beta L} \left[ \frac{\sin(\beta L/2)}{\beta L/2} \right]^2 \]

first null at \( 2\pi \)

A matching section with an exponential impedance taper.
(a) Variation of impedance.
(b) Resulting reflection coefficient magnitude response.

A matching section with a triangular taper for \( d(\ln Z/Z_0/dz). \)
(a) Variation of impedance.
(b) Resulting reflection coefficient magnitude response.