Chapter 2 Second-Order Linear ODEs

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Summary of Chapter 2
2.1 Homogeneous Linear ODEs of Second Order

If \( r(x) = 0 \) (that is, \( r(x) = 0 \) for all \( x \) considered; read “\( r(x) \) is identically zero”), then (1) reduces to

\[
y'' + p(x)y' + q(x)y = 0
\]

and is called **homogeneous**. If \( r(x) \neq 0 \), then (1) is called **nonhomogeneous**. This is similar to Sec. 1.5.

For instance, a nonhomogeneous linear ODE is

\[
y'' + 25y = e^{-x} \cos x,
\]
and a homogeneous linear ODE is

$$xy'' + y' + xy = 0.$$  

in standard form $$y'' + \frac{1}{x} y' + y = 0.$$  

An example of a nonlinear ODE is

$$y''y + y'^2 = 0.$$  

The functions $p$ and $q$ in (1) and (2) are called the **coefficients** of the ODEs.

**Solutions** are defined similarly as for first-order ODEs in Chap. 1. A function

$$y = h(x)$$
is called a solution of a (linear or nonlinear) second-order ODE on some open interval $I$ if $h$ is defined and twice differentiable throughout that interval and is such that the ODE becomes an identity if we replace the unknown $y$ by $h$, the derivative $y'$ by $h'$, and the second derivative $y''$ by $h''$. Examples are given below.
Homogeneous Linear ODEs: Superposition Principle

**EXAMPLE 1** Homogeneous Linear ODEs: Superposition of Solutions

The functions $y = \cos x$ and $y = \sin x$ are solutions of the homogeneous linear ODE

$$y'' + y = 0$$

for all $x$. We verify this by differentiation and substitution. We obtain $(\cos x)'' = -\cos x$; hence

$$y'' + y (\cos x)'' + \cos x = -\cos x + \cos x = 0.$$
Similarly for $y = \sin x$ (verify!). We can go an important step further. We multiply $\cos x$ by any constant, for instance, 4.7, and $\sin x$ by, say, $-2$, and take the sum of the results, claiming that it is a solution. Indeed, differentiation and substitution gives

$$(4.7 \cos x - 2 \sin x)'' + (4.7 \cos x - 2 \sin x)$$

$$= -4.7 \cos x + 2 \sin x + 4.7 \cos x - 2 \sin x = 0.$$
Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), any linear combination of two solutions on an open interval I is again a solution of (2) on I. In particular, for such an equation, sums and constant multiples of solutions are again solutions.
PROOF

Let \( y_1 \) and \( y_2 \) be solutions of (2) on \( I \). Then by substituting \( y = c_1 y_1 + c_2 y_2 \) and its derivatives into (2), and using the familiar rule \((c_1 y_1 + c_2 y_2)' = c_1 y_1' + c_2 y_2'\), etc., we get

\[
y'' + py' + qy = (c_1 y_1 + c_2 y_2)'' + p(c_1 y_1 + c_2 y_2)' + q(c_1 y_1 + c_2 y_2)
\]

\[
= c_1 y_1'' + c_2 y_2'' + p(c_1 y_1' + c_2 y_2') + q(c_1 y_1 + c_2 y_2)
\]

\[
= c_1(y_1'' + py_1' + qy_1) + c_2(y_2'' + py_2' + qy_2) = 0,
\]

since in the last line, \((\cdot \cdot \cdot) = 0\) because \( y_1 \) and \( y_2 \) are solutions, by assumption. This shows that \( y \) is a solution of (2) on \( I \).
EXAMPLE 2 A Nonhomogeneous Linear ODE

Verify by substitution that the functions $y = 1 + \cos x$ and $y = 1 + \sin x$ are solutions of the nonhomogeneous linear ODE

$$y'' + y = 1,$$

but their sum is not a solution. Neither is, for instance, $2(1 + \cos x)$ or $5(1 + \sin x)$. 
EXAMPLE 3 A Nonlinear ODE

Verify by substitution that the functions $y = x^2$ and $y = 1$ are solutions of the nonlinear ODE

$$y''y - xy = 0,$$

but their sum is not a solution. Neither is $-x^2$, so you cannot even multiply by $-1$!
Initial Value Problem. Basis. General Solution

For a second-order homogeneous linear ODE (2) an initial value problem consists of (2) and two initial conditions

\[ y(x_0) = K_0, \quad y'(x_0) = K_1. \]

These conditions prescribe given values \( K_0 \) and \( K_1 \) of the solution and its first derivative (the slope of its curve) at the same given \( x = x_0 \) in the open interval considered.
The conditions (4) are used to determine the two arbitrary constants \( c_1 \) and \( c_2 \) in a **general solution** (5)

\[
y = c_1 y_1 + c_2 y_2
\]

of the ODE; here, \( y_1 \) and \( y_2 \) are suitable solutions of the ODE.
EXAMPLE 4  Initial Value Problem

Solve the initial value problem

\[ y'' + y = 0, \quad y(0) = 3.0, \quad y'(0) = -0.5. \]

**Solution. Step 1. General solution.** The functions \( \cos x \) and \( \sin x \) are solutions of the ODE (by Example 1), and we take

\[ y = c_1 \cos x + c_2 \sin x. \]

This will turn out to be a general solution as defined below.
Step 2. Particular solution. We need the derivative \( y' = -c_1 \sin x + c_2 \cos x \). From this and the initial values we obtain, since \( \cos 0 = 1 \) and \( \sin 0 = 0 \),

\[
y(0) = c_1 = 3.0 \quad \text{and} \quad y'(0) = c_2 = -0.5.
\]

This gives as the solution of our initial value problem the particular solution

\[
y = 3.0 \cos x - 0.5 \sin x.
\]

Figure 28 shows that at \( x = 0 \) it has the value 3.0 and the slope \(-0.5\), so that its tangent intersects the \( x \)-axis at \( x = 3.0/0.5 = 6.0 \). (The scales on the axes differ!)
Fig. 28. Particular solution and initial tangent in Example 4
DEFINITION

General Solution, Basis, Particular Solution

A **general solution** of an ODE (2) on an open interval $I$ is a solution (5) in which $y_1$ and $y_2$ are solutions of (2) on $I$ that are not proportional, and $c_1$ and $c_2$ are arbitrary constants. These $y_1$, $y_2$ are called a **basis** (or a **fundamental system**) of solutions of (2) on $I$.

A **particular solution** of (2) on $I$ is obtained if we assign specific values to $c_1$ and $c_2$ in (5).
Namely, two functions $y_1$ and $y_2$ are called **linearly independent** on an interval $I$ where they are defined if

$$k_1 y_1(x) + k_2 y_2(x) = 0$$

everywhere on $I$ implies

$$k_1 = 0 \text{ and } k_2 = 0.$$
And $y_1$ and $y_2$ are called **linearly dependent** on $l$ if (7) also holds for some constants $k_1, k_2$ not both zero. Then if $k_1 \neq 0$ or $k_2 \neq 0$, we can divide and see that $y_1$ and $y_2$ are proportional,

$$y_1 = -\frac{k_2}{k_1} y_2 \text{ or } y_2 = -\frac{k_1}{k_2} y_1.$$ 

In contrast, in the case of linear **independence** these functions are not proportional because then we cannot divide in (7). This gives the following
Basis (Reformulated)

A **basis** of solutions of (2) on an open interval $I$ is a pair of linearly independent solutions of (2) on $I$. 
EXAMPLE 5  Basis, General Solution, Particular Solution

\( \cos x \) and \( \sin x \) in Example 4 form a basis of solutions of the ODE \( y'' + y = 0 \) for all \( x \) because their quotient is \( \cot x \neq \text{const} \) (or \( \tan x \neq \text{const} \)). Hence \( y = c_1 \cos x + c_2 \sin x \) is a general solution. The solution \( y = 3.0 \cos x - 0.5 \sin x \) of the initial value problem is a particular solution.
EXAMPLE 6  Basis, General Solution, Particular Solution

Verify by substitution that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the ODE $y'' - y = 0$. Then solve the initial value problem

$$y - y = 0, \quad y(0) = 6, \quad y'(0) = -2.$$
\textbf{Solution.} \((e^x)'' - e^x = 0\) and \((e^{-x})'' - e^{-x} = 0\) shows that \(e^x\) and \(e^{-x}\) are solutions. They are not proportional, \(e^x/e^{-x} = e^{2x} \neq \text{const.}\) Hence \(e^x, e^{-x}\) form a basis for all \(x\). We now write down the corresponding general solution and its derivative and equate their values at 0 to the given initial conditions,

\[ y = c_1 e^x + c_2 e^{-x}, \quad y' = c_1 e^x - c_2 e^{-x}, \]

\[ y(0) = c_1 + c_2 = 6, \quad y'(0) = c_1 - c_2 = -2. \]

By addition and subtraction, \(c_1 = 2, c_2 = 4\), so that the \textit{answer} is \(y = 2e^x + 4e^{-x}\). This is the particular solution satisfying the two initial conditions.
Find a Basis if One Solution Is Known. Reduction of Order

**EXAMPLE 7** Reduction of Order if a Solution Is Known.

**Basis**

Find a basis of solutions of the ODE

\[(x^2 - x)y'' - xy' + y = 0.\]

**Solution.** Inspection shows that \(y_1 = x\) is a solution because \(y_1' = 1\) and \(y_1'' = 0\), so that the first term vanishes identically and the second and third terms cancel. The idea of the method is to substitute

\[y = uy_1 = ux, \quad y' = u'x + u, \quad y'' = u''x + 2u\]

into the ODE. This gives

\[(x^2 - x)(u''x + 2u') - x(u'x + u) + ux = 0.\]
ux and –xu cancel and we are left with the following ODE, which we divide by x, order, and simplify,

\[(x^2 - x)(u''x + 2u') - x^2 u' = 0,
\]

\[(x^2 - x)u'' + (x - 2)u' = 0.\]

This ODE is of first order in \(v = u'\), namely, \((x^2 - x)v' + (x - 2)v = 0.\) Separation of variables and integration gives

\[
\frac{dv}{v} = -\frac{x - 2}{x^2 - x} \, dx = \left(\frac{1}{x - 1} - \frac{2}{x}\right) \, dx,
\]

\[\ln |v| = \ln |x - 1| - 2 \ln |x| = \ln \frac{|x - 1|}{x^2}.\]
We need no constant of integration because we want to obtain a particular solution; similarly in the next integration. Taking exponents and integrating again, we obtain

\[ v = \frac{x - 1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \quad u = \int v \, dx = \ln |x| + \frac{1}{x}, \]

hence

\[ y_2 = ux = x \ln |x| + 1. \]

Since \( y_1 = x \) and \( y_2 = x \ln \frac{x}{\circ} + 1 \) are linearly independent (their quotient is not constant), we have obtained a basis of solutions, valid for all positive \( x \).
2.2 Homogeneous Linear ODEs with Constant Coefficients

We shall now consider second-order homogeneous linear ODEs whose coefficients $a$ and $b$ are constant,

\[(1) \quad y'' + ay' + by = 0.\]

These equations have important applications, especially in connection with mechanical and electrical vibrations, as we shall see in Secs. 2.4, 2.8, and 2.9.
How to solve (1)? We remember from Sec. 1.5 that the solution of the \textit{first-order} linear ODE with a constant coefficient $k$

$$y' + ky = 0$$

is an exponential function $y = ce^{-kx}$. This gives us the idea to try as a solution of (1) the function

(2) \quad y = e^{\lambda x}.
Substituting (2) and its derivatives

\[ y = \lambda e^{\lambda x} \quad \text{and} \quad y = \lambda^2 e^{\lambda x} \]

into our equation (1), we obtain

\[(\lambda^2 + a\lambda + b)e^{\lambda x} = 0.\]

Hence if \(\lambda\) is a solution of the important characteristic equation (or auxiliary equation)

(3) \(\lambda^2 + a\lambda + b = 0\)
then the exponential function (2) is a solution of the ODE (1). Now from elementary algebra we recall that the roots of this quadratic equation (3) are

\[ \lambda_1 = \frac{1}{2}(-a + \sqrt{a^2 - 4b}), \quad \lambda_2 = \frac{1}{2}(-a - \sqrt{a^2 - 4b}). \]

(3) and (4) will be basic because our derivation shows that the functions

\[ y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x} \]

are solutions of (1). Verify this by substituting (5) into (1).
From algebra we further know that the quadratic equation (3) may have three kinds of roots, depending on the sign of the discriminant $a^2 - 4b$, namely,

(Case I) \( \text{Two real roots if } a^2 - 4b > 0, \)

(Case II) \( \text{A real double root if } a^2 - 4b = 0, \)

(Case III) \( \text{Complex conjugate roots if } a^2 - 4b < 0. \)
Case I. Two Distinct Real Roots $\lambda_1$ and $\lambda_2$

In this case, a basis of solutions of (1) on any interval is

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

because $y_1$ and $y_2$ are defined (and real) for all $x$ and their quotient is not constant. The corresponding general solution is

$$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$
EXAMPLE 1  General Solution in the Case of Distinct Real Roots

We can now solve $y'' - y = 0$ in Example 6 of Sec. 2.1 systematically. The characteristic equation is $\lambda^2 - 1 = 0$. Its roots are $\lambda_1 = 1$ and $\lambda_2 = -1$. Hence a basis of solutions is $e^x$ and $e^{-x}$ and gives the same general solution as before,

$$y = c_1 e^x + c_2 e^{-x}.$$
EXAMPLE 2  Initial Value Problem in the Case of Distinct Real Roots

Solve the initial value problem

\[ y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5. \]

**Solution.** Step 1. General solution. The characteristic equation is

\[ \lambda^2 + \lambda - 2 = 0. \]

Its roots are

\[ \lambda_1 = \frac{1}{2}(-1 + \sqrt{9}) = 1 \quad \text{and} \quad \lambda_2 = \frac{1}{2}(-1 - \sqrt{9}) = -2 \]

so that we obtain the general solution

\[ y = c_1 e^x + c_2 e^{-2x}. \]
Step 2. Particular solution. Since $y'(x) = c_1 e^x - 2c_2 e^{-2x}$, we obtain from the general solution and the initial conditions

$$y(0) = c_1 + c_2 = 4,$$

$$y'(0) = c_1 - 2c_2 = -5.$$  

Hence $c_1 = 1$ and $c_2 = 3$. This gives the answer $y = e^x + 3e^{-2x}$. Figure 29 shows that the curve begins at $y = 4$ with a negative slope ($-5$, but note that the axes have different scales!), in agreement with the initial conditions.
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Fig. 29. Solution in Example 2
Case II. Real Double Root \( \lambda = -a/2 \)

If the discriminant \( a^2 - 4b \) is zero, we see directly from (4) that we get only one root, \( \lambda = \lambda_1 = \lambda_2 = -a/2 \), hence only one solution,

\[
y_1 = e^{-(a/2)x}.
\]

in the case of a double root of (3) a basis of solutions of (1) on any interval is

\[
e^{-ax/2}, \quad xe^{-ax/2}.
\]

The corresponding general solution is

(7) \[
y = (c_1 + c_2x)e^{-ax/2}.
\]
EXAMPLE 3  General Solution in the Case of a Double Root

The characteristic equation of the ODE \( y'' + 6y' + 9y = 0 \) is \( \lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0 \). It has the double root \( \lambda = -3 \). Hence a basis is \( e^{-3x} \) and \( xe^{-3x} \). The corresponding general solution is \( y = (c_1 + c_2x)e^{-3x} \).
EXAMPLE 4  Initial Value Problem in the Case of a Double Root

Solve the initial value problem

\[ y'' + y' + 0.25y = 0, \quad y(0) = 3.0, \quad y'(0) = -3.5. \]

Solution. The characteristic equation is \( \lambda^2 + \lambda + 0.25 = (\lambda + 0.5)^2 = 0 \). It has the double root \( \lambda = -0.5 \). This gives the general solution

\[ y = (c_1 + c_2x)e^{-0.5x}. \]
We need its derivative

\[ y' = c_2 e^{-0.5x} - 0.5(c_1 + c_2x)e^{-0.5x}. \]

From this and the initial conditions we obtain

\[ y(0) = c_1 = 3.0, \quad y'(0) = c_2 - 0.5c_1 = 3.5; \]

hence \( c_2 = -2. \)

The particular solution of the initial value problem is

\[ y = (3 - 2x)e^{-0.5x}. \] See Fig. 30.
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Fig. 30. Solution in Example 4
Case III. Complex Roots – $\frac{1}{2}a + i \, \omega$ and $\frac{1}{2}a - i \, \omega$

This case occurs if the discriminant $a^2 - 4b$ of the characteristic equation (3) is negative. In this case, the roots of (3) and thus the solutions of the ODE (1) come at first out complex. However, we show that from them we can obtain a basis of real solutions

$$ y_1 = e^{-ax/2} \cos \omega \, x, \quad y_2 = e^{-ax/2} \sin \omega \, x \quad (\omega > 0) $$
where \( \square^2 = b - 1/4a^2 \). It can be verified by substitution that these are solutions in the present case. We shall derive them systematically after the two examples by using the complex exponential function. They form a basis on any interval since their quotient \( \cot x \) is not constant. Hence a real general solution in Case III is

\[
y = e^{-ax/2} (A \cos \square x + B \sin \square x)
\]

\((A, B \text{ arbitrary})\).
EXAMPLE 5 Complex Roots. Initial Value Problem

Solve the initial value problem

\[ y'' + 0.4y' + 9.04 y = 0, \quad y(0) = 0, \quad y'(0) = 3. \]

**Solution.** **Step 1. General solution.** The characteristic equation is \( \lambda^2 + 0.4 \lambda + 9.04 = 0 \). It has the roots \(-0.2 \pm 3i\). Hence \( \omega = 3 \), and a general solution (9) is

\[ y = e^{-0.2x}(A \cos 3x + B \sin 3x). \]

**Step 2. Particular solution.** The first initial condition gives \( y(0) = A = 0 \). The remaining expression is \( y = Be^{-0.2x} \sin 3x \). We need the derivative (chain rule!)
\[ y' = B(-0.2e^{-0.2x}\sin 3x + 3e^{-0.2x}\cos 3x). \]

From this and the second initial condition we obtain $y'(0) = 3B = 3$. Hence $B = 1$. Our solution is

\[ y = e^{-0.2x}\sin 3x. \]

Figure 31 shows $y$ and the curves of $e^{-0.2x}$ and $-e^{-0.2x}$ (dashed), between which the curve of $y$ oscillates. Such “damped vibrations” (with $x = t$ being time) have important mechanical and electrical applications, as we shall soon see (in Sec. 2.4).
Fig. 31. Solution in Example 5
EXAMPLE 6 Complex Roots

A general solution of the ODE

\[ y'' + \omega^2 y = 0 \]  \hspace{1cm} (\omega \text{ constant, not zero})

is

\[ y = A \cos \omega x + B \sin \omega x. \]

With \( \omega = 1 \) this confirms Example 4 in Sec. 2.1.
# Chapter 2 Second-Order Linear ODEs

## Summary of Cases I–III

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<td>$y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$</td>
</tr>
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<td>II</td>
<td>Real double root $\lambda = -\frac{1}{2}a$</td>
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<td>$y = (c_1 + c_2 x)e^{-ax/2}$</td>
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<tr>
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<td>Complex conjugate $\lambda_1 = -\frac{1}{2}a + i\omega$, $\lambda_2 = -\frac{1}{2}a - i\omega$</td>
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<td>$y = e^{-ax/2}(A \cos \omega x + B \sin \omega x)$</td>
</tr>
</tbody>
</table>
Derivation in Case III.
Complex Exponential Function

If verification of the solutions in (8) satisfies you, skip the systematic derivation of these real solutions from the complex solutions by means of the complex exponential function $e^z$ of a complex variable $z = r + it$. We write $r + it$, not $x + iy$ because $x$ and $y$ occur in the ODE. The definition of $e^z$ in terms of the real functions $e^r$, $\cos t$, and $\sin t$ is

(10) $e^z = e^{r+it} = e^r e^{it} = e^r(\cos t + i \sin t)$. 

\( e^{it} = \cos t + i \sin t, \) called the **Euler formula**. Multiplication by \( e^r \) gives (10).

For later use we note that \( e^{-it} = \cos (-t) + i \sin (-t) \) \( \cos t - i \sin t, \) so that by addition and subtraction of this and (11),

\[
\cos t = \frac{1}{2} (e^{it} + e^{-it}), \quad \sin t = \frac{1}{2i} (e^{it} - e^{-it}).
\]
2.3 Differential Operators. *Optional*

**Example 1** Factorization, Solution of an ODE

Factor $P(D) = D^2 - 3D - 40I$ and solve $P(D)y = 0$.

**Solution.** $D^2 - 3D - 40I = (D - 8I)(D + 5I)$ because $l^2 = l$. Now $(D - 8I)y = y - 8y = 0$ has the solution $y_1 = e^{8x}$. Similarly, the solution of $(D + 5I)y = 0$ is $y_2 = e^{-5x}$. This is a basis of $P(D)y = 0$ on any interval. From the factorization we obtain the ODE, as expected,
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\[(D - 8I)(D + 5I)y = (D - 8I)(y' + 5y)\]
\[= D(y' + 5y) - 8(y' + 5y)\]
\[= y'' + 5y' - 8y' - 40y\]
\[= y'' - 3y' - 40y = 0.\]

Verify that this agrees with the result of our method in Sec. 2.2. This is not unexpected because we factored \(P(D)\) in the same way as the characteristic polynomial \(P(\lambda) = \lambda^2 - 3\lambda - 40.\)
2.4 Modeling: Free Oscillations  
(Mass–Spring System)

Linear ODEs with constant coefficients have important applications in mechanics, as we show now (and in Sec. 2.8), and in electric circuits (to be shown in Sec. 2.9). In this section we consider a basic mechanical system, a mass on an elastic spring (“mass-spring system,” Fig. 32), which moves up and down. Its model will be a homogeneous linear ODE.
Setting Up the Model

We take an ordinary spring that resists compression as well extension and suspend it vertically from a fixed support, as shown in Fig. 32. At the lower end of the spring we attach a body of mass $m$. We assume $m$ to be so large that we can neglect the mass of the spring. If we pull the body down a certain distance and then release it, it starts moving. We assume that it moves strictly vertically.
Fig. 32. Mechanical mass–spring system
How can we obtain the motion of the body, say, the displacement $y(t)$ as function of time $t$? Now this motion is determined by Newton’s second law

\begin{equation}
\text{Mass } \times \text{Acceleration} = my'' = \text{Force}
\end{equation}

where $y'' = \frac{d^2y}{dt^2}$ and “Force” is the resultant of all the forces acting on the body.

(For systems of units and conversion factors, see the inside of the front cover.)

We choose the **downward direction as the positive direction**, thus regarding downward forces as positive and upward forces as negative.
Consider Fig. 32. The spring is first unstretched. We now attach the body. This stretches the spring by an amount $s_0$ shown in the figure. It causes an upward force $F_0$ in the spring. Experiments show that $F_0$ is proportional to the stretch $s_0$, say,

\[ F_0 = -ks_0 \]  \hspace{1cm} \text{(Hooke’s law).}

$k (> 0)$ is called the 	extbf{spring constant} (or 	extit{spring modulus}). The minus sign indicates that $F_0$ points upward, in our negative direction. Stiff springs have large $k$. (Explain!)
The extension $s_0$ is such that $F_0$ in the spring balances the weight $W = mg$ of the body (where $g = 980 \text{ cm/sec}^2 = 32.17 \text{ ft/sec}^2$ is the gravitational constant). Hence $F_0 + W = -ks_0 + mg = 0$. These forces will not affect the motion. Spring and body are again at rest. This is called the \textbf{static equilibrium} of the system (Fig. 32b). We measure the displacement $y(t)$ of the body from this ‘equilibrium point’ as the origin $y = 0$, downward positive and upward negative.
From the position $y = 0$ we pull the body downward. This further stretches the spring by some amount $y > 0$ (the distance we pull it down). By Hooke’s law this causes an (additional) upward force $F_1$ in the spring,

$$F_1 = -ky.$$  

$F_1$ is a **restoring force.** It has the tendency to **restore** the system, that is, to pull the body back to $y = 0$.  

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**Chapter 2 Second-Order Linear ODEs**
Undamped System: ODE and Solution

Every system has damping—otherwise it would keep moving forever. But practically, the effect of damping may often be negligible, for example, for the motion of an iron ball on a spring during a few minutes. Then $F_1$ is the only force in (1) causing the motion. Hence (1) gives the model $my'' = -ky$ or

$$my'' + ky = 0. \tag{3}$$

We obtain as a general solution

$$y(t) = A \cos \omega_0 t \ B \sin \omega_0 t, \quad \omega_0 = \sqrt{\frac{k}{m}}. \tag{4}$$

The corresponding motion is called a **harmonic oscillation**.
Since the trigonometric functions in (4) have the period $2\pi/\omega_0$, the body executes $\omega_0/2\pi$ cycles per second. This is the frequency of the oscillation, which is also called the natural frequency of the system. It is measured in cycles per second. Another name for cycles/sec is hertz (Hz).

The sum in (4) can be combined into a phase-shifted cosine with amplitude $C = \sqrt{A^2 + B^2}$ and phase angle $\delta = \arctan (B/A)$,

\[ y(t) = C \cos (\omega_0 t - \delta). \]
Fig. 33. Harmonic oscillations

- Positive
- Zero
- Negative

Initial velocity
EXAMPLE 1  Undamped Motion. Harmonic Oscillation

If an iron ball of weight \( W = 98 \text{ nt} \) (about 22 lb) stretches a spring 1.09 m (about 43 in.), how many cycles per minute will this mass–spring system execute? What will its motion be if we pull down the weight an additional 16 cm (about 6 in.) and let it start with zero initial velocity?

Solution. Hooke’s law (2) with \( W \) as the force and 1.09 meter as the stretch gives \( W = 1.09k \); thus \( k = W/1.09 = 98/1.09 = 90 \text{ [kg/sec}^2\text{]} = 90 \text{ [nt/meter]} \). The mass is \( m = W/g = 98/9.8 = 10 \text{ [kg]} \). This gives the frequency \( \omega_0/(2 \pi) = (k/m)^{1/2}/(2 \pi) = 3/(2 \pi) = 0.48 \text{ [Hz]} = 29 \text{ [cycles/min]} \).
From (4) and the initial conditions, \( y(0) = A = 0.16 \) [meter] and \( y'(0) = \omega_0 B = 0 \). Hence the motion is

\[
y(t) = 0.16 \cos 3t \text{ [meter]} \quad \text{or} \quad 0.52 \cos 3t \text{ [ft]}
\]

If you have a chance of experimenting with a mass–spring system, don’t miss it. You will be surprised about the good agreement between theory and experiment, usually within a fraction of one percent if you measure carefully.

Fig. 34. Harmonic oscillation in Example 1
Damped System: ODE and Solutions

We now add a damping force

\[ F_2 = -cy' \]

to our model \( my'' = -ky \), so that we have \( my'' = -ky - cy' \) or

\[ (5) \quad my'' + cy' + ky = 0. \]

Physically this can be done by connecting the body to a dashpot; see Fig. 35. We assume this new force to be proportional to the velocity \( y' = dy/dt \), as shown. This is generally a good approximation, at least for small velocities.
Fig. 35. Damped system
c is called the **damping constant**. We show that c is positive. If at some instant, \( y' \) is positive, the body is moving downward (which is the positive direction). Hence the damping force \( F_2 = -cy' \), always acting *against* the direction of motion, must be an upward force, which means that it must be negative, \( F_2 = -cy' < 0 \), so that \(-c < 0 \) and \( c > 0 \). For an upward motion, \( y' < 0 \) and we have a downward \( F_2 = -cy' > 0 \); hence \(-c < 0 \) and \( c > 0 \), as before.
The ODE (5) is homogeneous linear and has constant coefficients. Hence we can solve it by the method in Sec. 2.2. The characteristic equation is (divide (5) by $m$)

$$\lambda^2 + \frac{c}{m} \lambda + \frac{k}{m} = 0.$$ 

By the usual formula for the roots of a quadratic equation we obtain, as in Sec. 2.2,

$$\lambda_1 = -\alpha + \beta, \quad \lambda_2 = -\alpha - \beta,$$

(6) \hspace{1cm} \text{where} \quad \alpha = \frac{c}{2m} \quad \text{and} \quad \beta = \frac{1}{2m} \sqrt{c^2 - 4mk}.$$
It is now most interesting that depending on the amount of damping (much, medium, or little) there will be three types of motion corresponding to the three Cases I, II, III in Sec. 2.2:

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Nature of Root</th>
<th>Classification</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case I</td>
<td>$c^2 &gt; 4mk$</td>
<td>Distinct real roots $\lambda_1, \lambda_2$</td>
<td>Overdamping</td>
</tr>
<tr>
<td>Case II</td>
<td>$c^2 = 4mk$</td>
<td>A real double root</td>
<td>Critical damping</td>
</tr>
<tr>
<td>Case III</td>
<td>$c^2 &lt; 4mk$</td>
<td>Complex conjugate roots</td>
<td>Underdamping</td>
</tr>
</tbody>
</table>
Discussion of the Three Cases

Case I. Overdamping

If the damping constant $c$ is so large that $c^2 > 4mk$, then $\lambda_1$ and $\lambda_2$ are distinct real roots. In this case the corresponding general solution of (5) is

$$y(t) = c_1 e^{(\lambda_1 - \lambda_2)t} + c_2 e^{(\lambda_1 + \lambda_2)t}.$$
We see that in this case, damping takes out energy so quickly that the body does not oscillate. For $t > 0$ both exponents in (7) are negative because $\alpha > 0$, $\beta > 0$, and $\beta^2 = \alpha^2 - k/m < \alpha^2$. Hence both terms in (7) approach zero as $t \to \infty$. Practically speaking, after a sufficiently long time the mass will be at rest at the static equilibrium position ($y = 0$). Figure 36 shows (7) for some typical initial conditions.
Fig. 36. Typical motions (7) in the overdamped case
(a) Positive initial displacement
(b) Negative initial displacement
Case II. Critical Damping

Critical damping is the border case between nonoscillatory motions (Case I) and oscillations (Case III). It occurs if the characteristic equation has a double root, that is, if $c^2 = 4mk$, so that $\beta = 0$, $\lambda_1 = \lambda_2 = -\alpha$. Then the corresponding general solution of (5) is

$$y(t) = (c_1 + c_2 t)e^{-\alpha t}.$$
This solution can pass through the equilibrium position \( y = 0 \) at most once because \( e^{-\alpha t} \) is never zero and \( c_1 + c_2 t \) can have at most one positive zero. If both \( c_1 \) and \( c_2 \) are positive (or both negative), it has no positive zero, so that \( y \) does not pass through 0 at all. Figure 37 shows typical forms of (8). Note that they look almost like those in the previous figure.
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Fig. 37. Critical damping [see (8)]
Case III. Underdamping

This is the most interesting case. It occurs if the damping constant $c$ is so small that $c^2 < 4mk$. Then in (6) is no longer real but pure imaginary, say,

(9) \[ \beta = i \omega^* \]

where

\[ \omega^* = \frac{1}{2m} \sqrt{4mk - c^2} = \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}} \quad (> 0). \]

(We write $\omega^*$ to reserve for driving and electromotive forces in Secs. 2.8 and 2.9.) The roots of the characteristic equation are now complex conjugate,

\[ \lambda_1 = -\beta + i \omega^*, \quad \lambda_2 = -\beta - i \omega^* \]
with $\alpha = c/(2m)$, as given in (6). Hence the corresponding general solution is

$$y(t) = e^{-\alpha t}(A \cos \omega_* t + B \sin \omega_* t)$$

$$= Ce^{-\alpha t} \cos (\omega_* t - \delta)$$

where $C^2 = A^2 + B^2$ and $\tan \delta = B/A$, as in (4*). This represents **damped oscillations**. Their curve lies between the dashed curves $y = Ce^{-\alpha t}$ and $y = -Ce^{-\alpha t}$ in Fig. 38, touching them when $\omega_* t - \delta$ is an integer multiple of $\pi$ because these are the points at which $\cos (\omega_* t - \delta)$ equals 1 or −1.
The frequency is \( \omega^*/(2\pi) \) Hz (hertz, cycles/sec). From (9) we see that the smaller \( c \) (> 0) is, the larger is \( \omega^* \) and the more rapid the oscillations become. If \( c \) approaches 0, then \( \omega^* \) approaches \( \omega_0 = (k/m)^{1/2} \), giving the harmonic oscillation (4), whose frequency \( \omega_0/(2\pi) \) is the natural frequency of the system.
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Fig. 38. Damped oscillation in Case III [see (10)]
EXAMPLE 2 The Three Cases of Damped Motion

How does the motion in Example 1 change if we change the damping constant $c$ to one of the following three values, with $y(0) = 0.16$ and $y'(0) = 0$ as before?

(I) $c = 100 \text{ kg/sec}$,  (II) $c = 60 \text{ kg/sec}$,  (III) $c = 10 \text{ kg/sec}$.

Solution. It is interesting to see how the behavior of the system changes due to the effect of the damping, which takes energy from the system, so that the oscillations decrease in amplitude (Case III) or even disappear (Cases II and I).
With $m = 10$ and $k = 90$, as in Example 1, the model is the initial value problem

$$10y'' + 100y' + 90y = 0, \quad y(0) = 0.16 \text{ [meter]}, \quad y'(0) = 0.$$ 

The characteristic equation is $10\lambda^2 + 100\lambda + 90 = 10(\lambda + 9)(\lambda + 1) = 0$. It has the roots $-9$ and $-1$. This gives the general solution

$$y = c_1 e^{-9t} + c_2 e^{-t}.$$ 

We also need $y' = -9c_1 e^{-9t} - c_2 e^{-t}$. 
The initial conditions give $c_1 + c_2 = 0.16$, $-9c_1 - c_2 = 0$. The solution is $c_1 = -0.02$, $c_2 = 0.18$. Hence in the overdamped case the solution is

$$y = -0.02e^{-9t} + 0.18e^{-t}.$$ 

It approaches 0 as $t \to \infty$. The approach is rapid; after a few seconds the solution is practically 0, that is, the iron ball is at rest.
(II) The model is as before, with $c = 60$ instead of 100. The characteristic equation now has the form $10 \lambda^2 + 60 \lambda + 90 = 10( \lambda + 3)^2 = 0$. It has the double root $-3$. Hence the corresponding general solution is

$$\ y = (c_1 + c_2 t)e^{-3t}.$$

We also need $y' = (c_2 - 3c_1 - 3c_2 t)e^{-3t}$.

The initial conditions give $y(0) = c_1 = 0.16$, $y'(0) = c_2 - 3c_1 = 0$, $c_2 = 0.48$. Hence in the critical case the solution is

$$\ y = (0.16 + 0.48t)e^{-3t}.$$

It is always positive and decreases to 0 in a monotone fashion.
The model now is $10y'' + 10y' + 90y = 0$. Since $c = 10$ is smaller than the critical $c$, we shall get oscillations. The characteristic equation is $10\lambda^2 + 10\lambda + 90 = 10[(\lambda + \frac{1}{2})^2 + 9 - \frac{1}{4}] = 0$. It has the complex roots [see (4) in Sec. 2.2 with $a = 1$ and $b = 9$]

$$\lambda = -0.5 \pm \sqrt{0.5^2 - 9} = -0.5 \pm 2.96i.$$ 

This gives the general solution

$$y \ e^{-0.5t}(A \cos 2.96t + B \sin 2.96t).$$

Thus $y(0) = A = 0.16$. We also need the derivative
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\[ y = e^{-0.5t}(-0.5A \cos 2.96t - 0.5B \sin 2.96t - 2.96A \sin 2.96t + 2.96B \cos 2.96t). \]

Hence \( y'(0) = -0.5A + 2.96B = 0, \)

\[ B = 0.5A/2.96 = 0.027. \]

This gives the solution

\[ y = e^{-0.5t}(0.16 \cos 2.96t + 0.027 \sin 2.96t) \]
\[ = 0.162e^{-0.5t} \cos (2.96t - 0.17). \]

We see that these damped oscillations have a smaller frequency than the harmonic oscillations in Example 1 by about 1% (since 2.96 is smaller than 3.00 by about 1%). Their amplitude goes to zero. See Fig. 39.
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Fig. 39. The three solutions in Example 2
2.5 Euler–Cauchy Equations

Euler–Cauchy equations are ODEs of the form

\[ x^2 y'' + axy' + by = 0 \]  \hspace{1cm} (1)

with given constants \( a \) and \( b \) and unknown \( y(x) \). We substitute

\[ y = x^m \]  \hspace{1cm} (2)

and its derivatives \( y' = mx^{m-1} \) and \( y'' = m(m - 1)x^{m-2} \) into (1).

This gives

\[ x^2 m(m - 1)x^{m-2} + axmx^{m-1} + bx^m = 0. \]
We now see that (2) was a rather natural choice because we have obtained a common factor $x^m$. Dropping it, we have the auxiliary equation $m(m - 1) + am + b = 0$ or

\[ m^2 + (a - 1)m + b = 0. \]  

(Note: $a - 1$, not $a$.) Hence $y = x^m$ is a solution of (1) if and only if $m$ is a root of (3).
Case I. If the roots $m_1$ and $m_2$ are real and different, then solutions are

$$y_1(x) = x^{m_1} \quad \text{and} \quad y_2(x) = x^{m_2}$$

They are linearly independent since their quotient is not constant. Hence they constitute a basis of solutions of (1) for all $x$ for which they are real. The corresponding general solution for all these $x$ is

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

(c_1, c_2 arbitrary).
EXAMPLE 1  General Solution in the Case of Different Real Roots

The Euler–Cauchy equation

\[ x^2 y'' + 1.5xy' - 0.5y = 0 \]

has the auxiliary equation

\[ m^2 + 0.5m - 0.5 = 0. \] (Note: 0.5, not 1.5!)

The roots are 0.5 and –1. Hence a basis of solutions for all positive \( x \) is \( y_1 = x^{0.5} \) and \( y_2 = 1/x \) and gives the general solution

\[ y = c_1 \sqrt{x} + \frac{c_2}{x} \] for \( x > 0 \).
Case II. Equation (4) shows that the auxiliary equation (3) has a double root $m_1 = 1/2(1 - a)$ if and only if $(1 - a)^2 - 4b = 0$. The Euler–Cauchy equation (1) then has the form

\[ x^2y'' + axy' + \frac{1}{4}(1 - a)^2y = 0. \]

In this “critical case,” a basis of solutions for positive $x$ is $y_1 = x^m$ and $y_2 = x^m \ln x$, where $m = 1/2(1 - a)$. Linear independence follows from the fact that the quotient of these solutions is not constant. Hence, for all $x$ for which $y_1$ and $y_2$ are defined and real, a general solution is

\[ y = (c_1 + c_2 \ln x)x^m, \quad m = 1/2(1 - a) \]
EXAMPLE 2 General Solution in the Case of a Double Root

The Euler–Cauchy equation $x^2 y'' - 5xy' + 9y = 0$ has the auxiliary equation $m^2 - 6m + 9 = 0$. It has the double root $m = 3$, so that a general solution for all positive $x$ is

$$y = (c_1 + c_2 \ln x) x^3.$$
Case III. The case of complex roots is of minor practical importance, and it suffices to present an example that explains the derivation of real solutions from complex ones.
EXAMPLE 3  Real General Solution in the Case of Complex Roots

The Euler–Cauchy equation

\[ x^2 y'' + 0.6xy' + 16.04y = 0 \]

has the auxiliary equation \( m^2 - 0.4m + 16.04 = 0 \). The roots are complex conjugate, \( m_1 = 0.2 + 4i \) and \( m_2 = 0.2 - 4i \), where \( i = (-1)^{1/2} \). (We know from algebra that if a polynomial with real coefficients has complex roots, these are always conjugate.) Now use the trick of writing \( x = e^{\ln x} \) and obtain

\[
\begin{align*}
    x^{m_1} &= x^{0.2+4i} = x^{0.2}(e^{\ln x})^{4i} = x^{0.2}e^{(4 \ln x)i}, \\
    x^{m_2} &= x^{0.2-4i} = x^{0.2}(e^{\ln x})^{-4i} = x^{0.2}e^{-(4 \ln x)i}.
\end{align*}
\]
Next apply Euler’s formula (11) in Sec. 2.2 with \( t = 4 \ln x \) to these two formulas. This gives

\[
\begin{align*}
x^m_1 &= x^{0.2}[\cos (4 \ln x) + i \sin (4 \ln x)], \\
x^m_2 &= x^{0.2}[\cos (4 \ln x) - i \sin (4 \ln x)].
\end{align*}
\]

Add these two formulas, so that the sine drops out, and divide the result by 2. Then subtract the second formula from the first, so that the cosine drops out, and divide the result by \( 2i \). This yields

\[
x^{0.2} \cos (4 \ln x) \quad \text{and} \quad x^{0.2} \sin (4 \ln x)
\]
respectively. By the superposition principle in Sec. 2.2 these are solutions of the Euler–Cauchy equation (1). Since their quotient \( \cot (4 \ln x) \) is not constant, they are linearly independent. Hence they form a basis of solutions, and the corresponding real general solution for all positive \( x \) is

\[
y = x^{0.2}[A \cos (4 \ln x) + B \sin (4 \ln x)].
\]

Figure 47 shows typical solution curves in the three cases discussed, in particular the basis functions in Examples 1 and 3.
Fig. 47. Euler–Cauchy equations
EXAMPLE 4 Boundary Value Problem.  
Electric Potential Field Between Two Concentric Spheres

Find the electrostatic potential \( \nu = \nu(r) \) between two concentric spheres of radii \( r_1 = 5 \text{ cm} \) and \( r_2 = 10 \text{ cm} \) kept at potentials \( \nu_1 = 110 \text{ V} \) and \( \nu_2 = 0 \), respectively.

*Physical Information.* \( \nu(r) \) is a solution of the Euler–Cauchy equation \( r \nu'' + 2 \nu' = 0 \), where \( \nu' = dv/dr \).
**Solution.** The auxiliary equation is \( m^2 + m = 0 \). It has the roots 0 and -1. This gives the general solution \( v(r) = c_1 + c_2/r \). From the “boundary conditions” (the potentials on the spheres) we obtain

\[
\begin{align*}
v(5) &= c_1 + \frac{c_2}{5} = 110, \\
v(10) &= c_1 + \frac{c_2}{10} = 0.
\end{align*}
\]

By subtraction, \( c_2/10 = 110, \) \( c_2 = 1100 \). From the second equation, \( c_1 = -c_2/10 = -110 \). Answer: \( v(r) = -110 + 1100/r \) V. Figure 48 shows that the potential is not a straight line, as it would be for a potential between two parallel plates. For example, on the sphere of radius 7.5 cm it is not \( 110/2 = 55 \) V, but considerably less. (What is it?)
Fig. 48. Potential $v(r)$ in Example 4
2.6 Existence and Uniqueness of Solutions. Wronskian

In this section we shall discuss the general theory of homogeneous linear ODEs

(1) \[ y'' + p(x)y' + q(x)y = 0 \]

with continuous, but otherwise arbitrary variable coefficients \( p \) and \( q \). This will concern the existence and form of a general solution of (1) as well as the uniqueness of the solution of initial value problems consisting of such an ODE and two initial conditions

(2) \[ y(x_0) = K_0, \quad y'(x_0) = K_1 \]

with given \( x_0, K_0, \) and \( K_1 \).
Existence and Uniqueness Theorem for Initial Value Problems

If \( p(x) \) and \( q(x) \) are continuous functions on some open interval \( I \) (see Sec. 1.1) and \( x_0 \) is in \( I \), then the initial value problem consisting of (1) and (2) has a unique solution \( y(x) \) on the interval \( I \).
THEOREM 2

Linear Dependence and Independence of Solutions

Let the ODE (1) have continuous coefficients \( p(x) \) and \( q(x) \) on an open interval \( I \). Then two solutions \( y_1 \) and \( y_2 \) of (1) on \( I \) are linearly dependent on \( I \) if and only if their "Wronskian"

\[
W(y_1, y_2) = y_1 y'_2 - y_2 y'_1
\]

is 0 at some \( x_0 \) in \( I \). Furthermore, if \( W = 0 \) at an \( x = x_0 \) in \( I \), then \( W \not\equiv 0 \) on \( I \); hence if there is an \( x_1 \) in \( I \) at which \( W \) is not 0, then \( y_1, y_2 \) are linearly independent on \( I \).
Remark. Determinants. Students familiar with second-order determinants may have noticed that

\[
W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'.
\]

This determinant is called the \textit{Wronski determinant} or, briefly, the \textbf{Wronskian}, of two solutions \(y_1\) and \(y_2\) of (1), as has already been mentioned in (6). Note that its four entries occupy the same positions as in the linear system (7).
EXAMPLE 1 Illustration of Theorem 2

The functions \( y_1 = \cos \omega x \) and \( y_2 = \sin \omega x \) are solutions of \( y'' + \omega^2 y = 0 \). Their Wronskian is

\[
W(\cos \omega x, \sin \omega x) = \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix}
\]

\[
= y_1 y_2' - y_2 y_1' = \omega \cos^2 \omega x + \omega \sin^2 \omega x = \omega.
\]

Theorem 2 shows that these solutions are linearly independent if and only if \( \omega \neq 0 \). Of course, we can see this directly from the quotient \( y_2/y_1 = \tan \omega x \). For \( \omega = 0 \) we have \( y_2 \equiv 0 \), which implies linear dependence (why?).
EXAMPLE 2 Illustration of Theorem 2 for a Double Root

A general solution of \( y'' - 2y' + y = 0 \) on any interval is \( y = (c_1 + c_2x)e^x \). (Verify!). The corresponding Wronskian is not 0, which shows linear independence of \( e^x \) and \( xe^x \) on any interval. Namely,

\[
W(x, xe^x) = \begin{vmatrix} e^x & xe^x \\ xe^x & (x + 1)e^x \end{vmatrix} = (x + 1)e^{2x} - xe^{2x} = e^{2x} \neq 0.
\]
A General Solution of (1) Includes All Solutions

THEOREM 3

Existence of a General Solution

If \( p(x) \) and \( q(x) \) are continuous on an open interval \( I \), then (1) has a general solution on \( I \).
A General Solution Includes All Solutions

If the ODE (1) has continuous coefficients $p(x)$ and $q(x)$ on some open interval $I$, then every solution $y = Y(x)$ of (1) on $I$ is of the form

\[ Y(x) = C_1 y_1(x) + C_2 y_2(x) \]

where $y_1, y_2$ is any basis of solutions of (1) on $I$ and $C_1, C_2$ are suitable constants.

Hence (1) does not have singular solutions (that is, solutions not obtainable from a general solution).
2.7 Nonhomogeneous ODEs

Method of Undetermined Coefficients

In this section we proceed from homogeneous to nonhomogeneous linear ODEs

\[ y'' + p(x)y' + q(x)y = r(x) \]

where \( r(x) \neq 0 \). We shall see that a “general solution” of (1) is the sum of a general solution of the corresponding homogeneous ODE

\[ y'' + p(x)y' + q(x)y = 0 \]

and a “particular solution” of (1). These two new terms “general solution of (1)” and “particular solution of (1)” are defined as follows.
A general solution of the nonhomogeneous ODE (1) on an open interval $I$ is a solution of the form

$$y(x) = y_h(x) + y_p(x);$$

here, $y_h = c_1y_1 + c_2y_2$ is a general solution of the homogeneous ODE (2) on $I$ and $y_p$ is any solution of (1) on $I$ containing no arbitrary constants.

A particular solution of (1) on $I$ is a solution obtained from (3) by assigning specific values to the arbitrary constants $c_1$ and $c_2$ in $y_h$. 
THEOREM 1

Relations of Solutions of (1) to Those of (2)

a) The sum of a solution $y$ of (1) on some open interval $I$ and a solution $y"$ of (2) on $I$ is a solution of (1) on $I$. In particular, (3) is a solution of (1) on $I$.

b) The difference of two solutions of (1) on $I$ is a solution of (2) on $I$. 

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A General Solution of a Nonhomogeneous ODE Includes All Solutions

If the coefficients $p(x)$, $q(x)$, and the function $r(x)$ in (1) are continuous on some open interval $I$, then every solution of (1) on $I$ is obtained by assigning suitable values to the arbitrary constants $c_1$ and $c_2$ in a general solution (3) of (1) on $I$. 
Method of Undetermined Coefficients

The method of undetermined coefficients is suitable for linear ODEs with \textit{constant coefficients} \(a\) and \(b\)

\[
y'' + ay' + by = r(x)
\]

when \(r(x)\) is an exponential function, a power of \(x\), a cosine or sine, or sums or products of such functions. These functions have derivatives similar to \(r(x)\) itself. This gives the idea. We choose a form for \(y_p\) similar to \(r(x)\), but with unknown coefficients to be determined by substituting that \(y_p\) and its derivatives into the ODE. Table 2.1 shows the choice of \(y_p\) for practically important forms of \(r(x)\). Corresponding rules are as follows.
Choice Rules for the Method of Undetermined Coefficients

(a) **Basic Rule.** If $r(x)$ in (4) is one of the functions in the first column in Table 2.1, choose $y_p$ in the same line and determine its undetermined coefficients by substituting $y_p$ and its derivatives into (4).

(b) **Modification Rule.** If a term in your choice for $y_p$ happens to be a solution of the homogeneous ODE corresponding to (4), multiply your choice of $y_p$ by $x$ (or by $x^2$ if this solution corresponds to a double root of the characteristic equation of the homogeneous ODE).
(c) **Sum Rule.** If \( r(x) \) is a sum of functions in the first column of Table 2.1, choose for \( y_p \) the sum of the functions in the corresponding lines of the second column.

The Basic Rule applies when \( r(x) \) is a single term. The Modification Rule helps in the indicated case, and to recognize such a case, we have to solve the homogeneous ODE first. The Sum Rule follows by noting that the sum of two solutions of (1) with \( r = r_1 \) and \( r = r_2 \) (and the same left side!) is a solution of (1) with \( r = r_1 + r_2 \). (Verify!)
**Table 2.1** Method of Undetermined Coefficients

<table>
<thead>
<tr>
<th>Term in $r(x)$</th>
<th>Choice for $y_p(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$ke^{\gamma x}$</td>
<td>$Ce^{\gamma x}$</td>
</tr>
<tr>
<td>$kx^n$ $(n = 0, 1, \cdots)$</td>
<td>$K_nx^n + K_{n-1}x^{n-1} + \cdots + K_1x + K_0$</td>
</tr>
<tr>
<td>$k \cos \omega x$</td>
<td>${ K \cos \omega x + M \sin \omega x }$</td>
</tr>
<tr>
<td>$k \sin \omega x$</td>
<td></td>
</tr>
<tr>
<td>$ke^{\alpha x} \cos \omega x$</td>
<td>$e^{\alpha x}(K \cos \omega x + M \sin \omega x)$</td>
</tr>
<tr>
<td>$ke^{\alpha x} \sin \omega x$</td>
<td></td>
</tr>
</tbody>
</table>
**EXAMPLE 1**  Application of the Basic Rule (a)

Solve the initial value problem

\( y'' + y = 0.001x^2, \quad y(0) = 0, \ y'(0) = 1.5. \)

*Solution.*  **Step 1. General solution of the homogeneous ODE.** The ODE \( y'' + y = 0 \) has the general solution

\[ y_h = A \cos x + B \sin x. \]

**Step 2. Solution \( y_p \) of the nonhomogeneous ODE.** We first try \( y_p = Kx^2 \). Then \( y''_p = 2K \). By substitution, \( 2K + Kx^2 = 0.001x^2 \). For this to hold for all \( x \), the coefficient of each power of \( x \) (\( x^2 \) and \( x^0 \)) must be the same on both sides; thus \( K = 0.001 \) and \( 2K = 0 \), a contradiction.
The second line in Table 2.1 suggests the choice

\[ y_p = K_2 x^2 + K_1 x + K_0. \]

Then

\[ y''_p + y_p = 2K_2 + K_2 x^2 + K_1 x + K_0 = 0.001 x^2. \]

Equating the coefficients of \( x^2, x, x^0 \) on both sides, we have \( K_2 = 0.001, K_1 = 0, 2K_2 + K_0 = 0. \) Hence \( K_0 = -2K_2 = -0.002. \) This gives \( y_p = 0.001 x^2 - 0.002, \) and

\[ y = y_h + y_p = A \cos x + B \sin x + 0.001 x^2 - 0.002. \]
**Step 3. Solution of the initial value problem.** Setting $x = 0$ and using the first initial condition gives $y(0) = A - 0.002 = 0$, hence $A = 0.002$. By differentiation and from the second initial condition,

\[
y' = y'_h + y'_p = -A \sin x + B \cos x + 0.002x
\]

and \[y'(0) = B = 1.5.\]

This gives the answer (Fig. 49)

\[
y = 0.002 \cos x + 1.5 \sin x + 0.001x^2 - 0.002.
\]

Figure 49 shows $y$ as well as the quadratic parabola $y_p$ about which $y$ is oscillating, practically like a sine curve since the cosine term is smaller by a factor of about 1/1000.
Fig. 49. Solution in Example 1
EXAMPLE 2  Application of the Modification Rule (b)

Solve the initial value problem

(6)  \[ y'' + 3y' + 2.25y = -10 \ e^{-1.5x}, \quad y(0) = 1, \ y'(0) = 0. \]

**Solution.** Step 1. General solution of the homogeneous ODE. The characteristic equation of the homogeneous ODE is \( \lambda^2 + 3 \lambda + 2.25 = (\lambda + 1.5)^2 = 0. \) Hence the homogeneous ODE has the general solution

\[ y_h = (c_1 + c_2 x) e^{-1.5x}. \]
Step 2. Solution $y_p$ of the nonhomogeneous ODE. The function $e^{-1.5x}$ on the right would normally require the choice $Ce^{-1.5x}$. But we see from $y_h$ that this function is a solution of the homogeneous ODE, which corresponds to a double root of the characteristic equation. Hence, according to the Modification Rule we have to multiply our choice function by $x^2$. That is, we choose

$$y_p = Cx^2e^{-1.5x}.$$

Then

$$y'_p = C(2x - 1.5x^2)e^{-1.5x},$$

$$y''_p = C(2 - 3x - 3x + 2.25x^2)e^{-1.5x}.$$
We substitute these expressions into the given ODE and omit the factor $e^{-1.5x}$. This yields

$$C(2 - 6x + 2.25x^2) + 3C(2x - 1.5x^2) + 2.25Cx^2 = -10.$$  

Comparing the coefficients of $x^2$, $x$, $x^0$ gives $0 = 0$, $0 = 0$, $2C = -10$, hence $C = -5$. This gives the solution $y_p = -5x^2e^{-1.5x}$. Hence the given ODE has the general solution

$$y = y_h + y_p = (c_1 + c_2x)e^{-1.5x} - 5x^2e^{-1.5x}.$$
Step 3. Solution of the initial value problem. Setting $x = 0$ in $y$ and using the first initial condition, we obtain $y(0) = c_1 = 1$. Differentiation of $y$ gives

$$y' = (c_2 - 1.5c_1 - 1.5c_2x)e^{-1.5x} - 10xe^{-1.5x} + 7.5x^2e^{-1.5x}.$$

From this and the second initial condition we have $y'(0) = c_2 - 1.5c_1 = 0$. Hence $c_2 = 1.5c_1 = 1.5$. This gives the answer (Fig. 50)

$$y = (1 + 1.5x) e^{-1.5x} - 5x^2e^{-1.5x} = (1 + 1.5x - 5x^2)e^{-1.5x}.$$

The curve begins with a horizontal tangent, crosses the $x$-axis at $x = 0.6217$ (where $1 + 1.5x - 5x^2 = 0$) and approaches the axis from below as $x$ increases.
Fig. 50. Solution in Example 2
EXAMPLE 3  Application of the Sum Rule (c)

Solve the initial value problem

\[ y'' + 2y' + 5y = e^{0.5x} + 40 \cos 10x - 190 \sin 10x, \]
\[ y(0) = 0.16, \ y'(0) = 40.08. \]

Solution. Step 1. General solution of the homogeneous ODE. The characteristic equation

\[ \lambda^2 + 2\lambda + 5 = (\lambda + 1 + 2i)(\lambda + 1 - 2i) = 0 \]

shows that a real general solution of the homogeneous ODE is

\[ y_h = e^{-x} (A \cos 2x + B \sin 2x). \]
Step 2. Solution of the nonhomogeneous ODE.

We write \( y_p = y_{p1} + y_{p2} \), where \( y_{p1} \) corresponds to the exponential term and \( y_{p2} \) to the sum of the other two terms. We set

\[ y_{p1} = Ce^{0.5x}. \]

Then

\[ y'_{p1} = 0.5Ce^{0.5x} \quad \text{and} \quad y''_{p1} = 0.25Ce^{0.5x}. \]

Substitution into the given ODE and omission of the exponential factor gives

\[ (0.25 + 2 \cdot 0.5 + 5)C = 1, \]

hence \( C = 1/6.25 = 0.16 \), and \( y_{p1} = 0.16e^{0.5x} \).
We now set \( y_{p2} = K \cos 10x + M \sin 10x \), as in Table 2.1, and obtain

\[
y'_{p2} = -10K \sin 10x + 10M \cos 10x,
\]

\[
y''_{p2} = -100K \cos 10x - 100M \sin 10x.
\]

Substitution into the given ODE gives for the cosine terms and for the sine terms

\[-100K + 2 \cdot 10M + 5K = 40,\]

\[-100M - 2 \cdot 10K + 5M = 190.\]
or, by simplification,

\[-95K + 20M = 40, \quad -20K - 95M = 190.\]

The solution is \( K = 0, \ M = 2. \) Hence \( y_{p2} = 2 \sin 10x. \) Together,

\[ y = y_h + y_{p1} + y_{p2} \]

\[ = e^{-x}(A \cos 2x + B \sin 2x) + 0.16e^{0.5x} + 2 \sin 10x. \]

**Step 3. Solution of the initial value problem.** From \( y \) and the first initial condition, \( y(0) = A + 0.16 = 0.16, \) hence \( A = 0. \) Differentiation gives

\[ y' = e^{-x}(-A \cos 2x - B \sin 2x - 2A \sin 2x + 2B \cos 2x) \]

\[ + 0.08e^{0.5x} + 20 \cos 10x. \]
From this and the second initial condition we have $y'(0) = -A + 2B + 0.08 + 20 = 40.08$, hence $B = 10$. This gives the solution (Fig. 51)

$$y = 10e^{-x} \sin 2x + 0.16e^{0.5x} + 2 \sin 10x.$$  

The first term goes to 0 relatively fast. When $x = 4$, it is practically 0, as the dashed curves $10e^{-x} + 0.16e^{0.5x}$ show. From then on, the last term, $2 \sin 10x$, gives an oscillation about $0.16e^{0.5x}$, the monotone increasing dashed curve.
Fig. 51. Solution in Example 3
2.8 Modeling: Forced Oscillations.

Resonance

In Sec. 2.4 we considered vertical motions of a mass-spring system (vibration of a mass $m$ on an elastic spring, as in Figs. 32 and 52) and modeled it by the homogeneous linear ODE

$$my'' + cy' + ky = 0.$$ 

Here $y(t)$ as a function of time $t$ is the displacement of the body of mass $m$ from rest. These were free motions, that is, motions in the absence of external forces (outside forces) caused solely by internal forces, forces within the system. These are the force of inertia $my''$, the damping force $cy'$ (if $c > 0$), and the spring force $ky$ acting as a restoring force.
We now extend our model by including an external force, call it $r(t)$, on the right. Then we have

\[(2^*) \quad my'' + cy' + ky = r(t).\]

Mechanically this means that at each instant $t$ the resultant of the internal forces is in equilibrium with $r(t)$. The resulting motion is called a \textbf{forced motion} with \textbf{forcing function} $r(t)$, which is also known as \textbf{input} or \textbf{driving force}, and the solution $y(t)$ to be obtained is called the \textbf{output} or the \textbf{response of the system to the driving force}.
Of special interest are periodic external forces, and we shall consider a driving force of the form

\[ r(t) = F_0 \cos \omega t \quad (F_0 > 0, \ \omega > 0). \]

Then we have the nonhomogeneous ODE

(2) \[ my'' + cy' + ky = F_0 \cos \omega t. \]
Fig. 52. Mass on a spring

$$r(t) = F_0 \cos \omega t$$
Solving the Nonhomogeneous ODE (2)

From Sec. 2.7 we know that a general solution of (2) is the sum of a general solution $y_h$ of the homogeneous ODE (1) plus any solution $y_p$ of (2). To find $y_p$, we use the method of undetermined coefficients (Sec. 2.7), starting from

$$(3) \quad y_p(t) = a \cos \omega t + b \sin \omega t.$$

If we set $(k/m)^{1/2} = \omega_0 (> 0)$ as in Sec. 2.4, then $k = m \omega_0^2$ and we obtain
We thus obtain the general solution of the nonhomogeneous ODE (2) in the form

\[ y(t) = y_h(t) + y_p(t). \]

Here \( y_h \) is a general solution of the homogeneous ODE (1) and \( y_p \) is given by (3) with coefficients (5).
Case 1. Undamped Forced Oscillations. Resonance

If the damping of the physical system is so small that its effect can be neglected over the time interval considered, we can set $c = 0$. Then (5) reduces to $a = F_0/[m(\omega_0^2 - \omega^2)]$ and $b = 0$. Hence (3) becomes (use $\omega_0^2 = k/m$)

\[
y_p(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t = \frac{F_0}{k[1 - (\omega/\omega_0)^2]} \cos \omega t.
\]
Here we must assume that $\omega^2 \neq \omega_0^2$; physically, the frequency $\omega/(2\pi)$ [cycles/sec] of the driving force is different from the *natural frequency* $\omega_0/(2\pi)$ of the system, which is the frequency of the free undamped motion [see (4) in Sec. 2.4]. From (7) and from (4*) in Sec. 2.4 we have the general solution of the “undamped system”

$$y(t) = C \cos (\omega_0 t - \delta) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$  

*We see that this output is a superposition of two harmonic oscillations of the frequencies just mentioned.*
Resonance. We discuss (7). We see that the maximum amplitude of $y_p$ is (put $\cos \omega t = 1$)

$$a_0 = \frac{F_0}{k} \rho$$

where

$$\rho = \frac{1}{1 - (\omega/\omega_0)^2}.$$

$a_0$ depends on $\omega$ and $\omega_0$. If $\omega = \omega_0$, then $\rho$ and $a_0$ tend to infinity. This excitation of large oscillations by matching input and natural frequencies ($\omega = \omega_0$) is called resonance. $\rho$ is called the resonance factor (Fig. 53), and from (9) we see that $\rho/k = a_0/F_0$ is the ratio of the amplitudes of the particular solution $y_p$ and of the input $F_0 \cos \omega t$. We shall see later in this section that resonance is of basic importance in the study of vibrating systems.
In the case of resonance the nonhomogeneous ODE (2) becomes

\[ y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t. \]  

(10)

Then (7) is no longer valid, and from the Modification Rule in Sec. 2.7 we conclude that a particular solution of (10) is of the form

\[ y_p(t) = t(a \cos \omega_0 t + b \sin \omega_0 t). \]
Fig. 53. Resonance factor $\rho(\omega)$
By substituting this into (10) we find $a = 0$ and $b = \frac{F_0}{2m\omega_0}$. Hence (Fig. 54)

\begin{equation}
    y_p(t) = \frac{F_0}{2m\omega_0} t \sin \omega_0 t.
\end{equation}

We see that because of the factor $t$ the amplitude of the vibration becomes larger and larger. Practically speaking, systems with very little damping may undergo large vibrations that can destroy the system. We shall return to this practical aspect of resonance later in this section.
Chapter 2 Second-Order Linear ODEs

Fig. 54. Particular solution in the case of resonance
Beats. Another interesting and highly important type of oscillation is obtained if \( \omega \) is close to \( \omega_0 \). Take, for example, the particular solution [see (8)]

\[
y(t) = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t)
\]

\( (\omega \approx \omega_0) \).

Using (12) in App. 3.1, we may write this as

\[
y(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \left( \frac{\omega_0 + \omega}{2} t \right) \sin \left( \frac{\omega_0 - \omega}{2} t \right).
\]
Since $\omega$ is close to $\omega_0$, the difference $\omega_0 - \omega$ is small. Hence the period of the last sine function is large, and we obtain an oscillation of the type shown in Fig. 55, the dashed curve resulting from the first sine factor. This is what musicians are listening to when they *tune* their instruments.
Fig. 55. Forced undamped oscillation when the difference of the input and natural frequencies is small (“beats”)

Chapter 2 Second-Order Linear ODEs
Case 2. Damped Forced Oscillations

If the damping of the mass-spring system is not negligibly small, we have $c > 0$ and a damping term $cy'$ in (1) and (2). Then the general solution $y_h$ of the homogeneous ODE (1) approaches zero as $t$ goes to infinity, as we know from Sec. 2.4. Practically, it is zero after a sufficiently long time. Hence the “transient solution” (6) of (2), given by $y = y_h + y_p$, approaches the “steady-state solution” $y_p$. This proves the following.
THEOREM 1

Steady-State Solution

After a sufficiently long time the output of a damped vibrating system under a purely sinusoidal driving force [see (2)] will practically be a harmonic oscillation whose frequency is that of the input.
To study the amplitude of $y_p$ as a function of $\omega$, we write (3) in the form

$$y_p(t) = C^* \cos (\omega t - \eta).$$

$C^*$ is called the amplitude of $y_p$ and $\eta$ the phase angle or phase lag because it measures the lag of the output behind the input. According to (5), these quantities are

$$C^*(\omega) = \sqrt{a^2 + b^2} = \frac{F_0}{\sqrt{m^2(\omega_0^2 - \omega^2)^2 + \omega^2c^2}},$$

and

$$\tan \eta(\omega) = \frac{b}{a} = \frac{\omega c}{m(\omega_0^2 - \omega^2)}.$$
We see that $C^*(\omega_{\text{max}})$ is always finite when $c > 0$. Furthermore, since the expression

$$C^2 4m^2 \omega_0^2 - c^4 = c^2(4mk - c^2)$$

in the denominator of (16) decreases monotone to zero as $c^2 (< 2mk)$ goes to zero, the maximum amplitude (16) increases monotone to infinity, in agreement with our result in Case 1.
Figure 56 shows the amplification $C^*/F_0$ (ratio of the amplitudes of output and input) as a function of $\omega$ for $m = 1$, $k = 1$, hence $\omega_0 = 1$, and various values of the damping constant $c$.

Figure 57 shows the phase angle (the lag of the output behind the input), which is less than $\pi/2$ when $\omega < \omega_0$, and greater than $\pi/2$ for $\omega > \omega_0$. 
Fig. 56. Amplification $C^*/F_0$ as a function of $\omega$ for $m = 1$, $k = 1$, and various values of the damping constant $c$. 
Fig. 57. Phase lag $\eta$ as a function of $\omega$ for $m = 1, k = 1$, thus $\omega_0 = 1$, and various values of the damping constant $c$
2.9 Modeling: Electric Circuits

We have just seen that linear ODEs have important applications in mechanics. Similarly, they are models of electric circuits, as they occur as portions of large networks in computers and elsewhere. The circuits we shall consider here are basic building blocks of such networks. They contain three kinds of components, namely, resistors, inductors, and capacitors. Figure 60 shows such an \textbf{RLC-circuit}, as they are called. In it a resistor of resistance $R \ \Omega$ (ohms), an inductor of inductance $L \ \text{H}$ (henrys), and a capacitor of capacitance $C \ \text{F}$ (farads) are wired in series as shown, and connected to an electromotive force $E(t) \ \text{V}$ (volts) (a generator, for instance), sinusoidal as in Fig. 60, or of some other kind. $R$, $L$, $C$, and $E$ are given and we want to find the current $I(t) \ \text{A}$ (amperes) in the circuit.
An ODE for the current \( I(t) \) in the \( RLC \)-circuit in Fig. 60 is obtained from the following law.

\[ E(t) = E_0 \sin \omega t \]

**Fig. 60.** \( RLC \)-circuit
Kirchhoff’s Voltage Law (KVL). The voltage (the electromotive force) impressed on a closed loop is equal to the sum of the voltage drops across the other elements of the loop.

In Fig. 60 the circuit is a closed loop, and the impressed voltage $E(t)$ equals the sum of the voltage drops across the three elements $R$, $L$, $C$ of the loop.
Voltage Drops. Experiments show that a current $I$ flowing through a resistor, inductor or capacitor causes a voltage drop (voltage difference, measured in volts) at the two ends; these drops are

$R I$ (Ohm’s law) Voltage drop for a resistor of resistance $R$ ohms ($\Omega$),

$LI' = L(dl/dt)$ Voltage drop for an inductor of inductance $L$ henrys (H),

$Q/C$ Voltage drop for a capacitor of capacitance $C$ farads (F).
Here $Q$ coulombs is the charge on the capacitor, related to the current by

$$I(t) = \frac{dQ}{dt}, \quad \text{equivalently,} \quad Q(t) = \int I(t) \, dt.$$  

This is summarized in Fig. 61.

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Notation</th>
<th>Unit</th>
<th>Voltage Drop</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ohm’s resistor</td>
<td>$R$</td>
<td>Ohm’s resistance</td>
<td>ohms ($\Omega$)</td>
<td>$RI$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$L$</td>
<td>Inductance</td>
<td>henrys (H)</td>
<td>$L \frac{dI}{dt}$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$C$</td>
<td>Capacitance</td>
<td>farads (F)</td>
<td>$Q/C$</td>
</tr>
</tbody>
</table>

**Fig. 61.** Elements in an $RLC$-circuit
According to KVL we thus have in Fig. 60 for an $RLC$-circuit with electromotive force $E(t) = E_0 \sin \omega t$ ($E_0$ constant) as a model the "integro-differential equation"

$$LI' + RI + \frac{1}{C} \int I \, dt = E(t) = E_0 \sin \omega t.$$  

(1')

To get rid of the integral, we differentiate (1) with respect to $t$, obtaining

$$LI'' + RI' + \frac{1}{C} I = E'(t) = E_0 \omega \cos \omega t.$$  

(1)
This shows that the current in an \( RLC \)-circuit is obtained as the solution of this nonhomogeneous second-order ODE (1) with constant coefficients.

From (1'), using \( I = Q' \), hence \( I = Q'' \), we also have directly

\[ LQ'' + RQ' + \frac{1}{C} Q = E_0 \sin \omega t. \]  

But in most practical problems the current \( I(t) \) is more important than the charge \( Q(t) \), and for this reason we shall concentrate on (1) rather than on (1'').
Solving the ODE (1) for the Current. 

Discussion of Solution

A general solution of (1) is the sum \( I = I_h + I_p \), where \( I_h \) is a general solution of the homogeneous ODE corresponding to (1) and \( I_p \) is a particular solution of (1). We first determine \( I_p \) by the method of undetermined coefficients, proceeding as in the previous section. We substitute

\[
I_p = a \cos \omega t + b \sin \omega t
\]

\[
I'_p = \omega (-a \sin \omega t + b \cos \omega t)
\]

\[
I''_p = \omega^2 (-a \cos \omega t - b \sin \omega t)
\]
into (1). Then we collect the cosine terms and equate them to $E_0 \cos \omega t$ on the right, and we equate the sine terms to zero because there is no sine term on the right,

$$L \omega^2 (-a) + R \omega b + a/C = E_0 \cos \omega t \quad \text{(Cosine terms)}$$

$$L \omega^2 (-b) + R \omega (-a) + b/C = 0 \quad \text{(Sine terms)}.$$

To solve this system for $a$ and $b$, we first introduce a combination of $L$ and $C$, called the reactance

$$S = \omega L - \frac{1}{\omega C} \quad (3)$$
In any practical case the resistance $R$ is different from zero, so that we can solve for $a$ and $b$,

$$a = \frac{-E_0 S}{R^2 + S^2}, \quad b = \frac{E_0 R}{R^2 + S^2}.$$  \hspace{1cm} (4)

Equation (2) with coefficients $a$ and $b$ given by (4) is the desired particular solution $I_p$ of the nonhomogeneous ODE (1) governing the current $I$ in an $RLC$-circuit with sinusoidal electromotive force.
Using (4), we can write $I_p$ in terms of “physically visible” quantities, namely, amplitude $I_0$ and phase lag $\theta$ of the current behind the electromotive force, that is,

$$I_p(t) = I_0 \sin (\theta t - \theta)$$

where [see (14) in App. A3.1]

$$I_0 = \sqrt{a^2 + b^2} = \frac{E_0}{\sqrt{R^2 + S^2}} , \quad \tan \theta = -\frac{a}{b} = \frac{S}{R} .$$

The quantity $(R^2 + S^2)^{1/2}$ is called the impedance. Our formula shows that the impedance equals the ratio $E_0/I_0$. This is somewhat analogous to $E/I = R$ (Ohm’s law).
A general solution of the homogeneous equation corresponding to (1) is

\[ I_h = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \]

where \( \lambda_1 \) and \( \lambda_2 \) are the roots of the characteristic equation

\[ \lambda^2 + \frac{R}{L} \lambda + \frac{1}{LC} = 0. \]

We can write these roots in the form \( \lambda_1 = -\alpha + \beta \) and \( \lambda_2 = -\alpha - \beta \), where

\[ \alpha = \frac{R}{2L}, \quad \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} = \frac{1}{2L} \sqrt{R^2 - \frac{4L}{C}}. \]
Now in an actual circuit, $R$ is never zero (hence $R > 0$). From this it follows that $I_h$ approaches zero, theoretically as $t \to \infty$, but practically after a relatively short time. (This is as for the motion in the previous section.) Hence the transient current $I = I_h + I_p$ tends to the steady-state current $I_p$, and after some time the output will practically be a harmonic oscillation, which is given by (5) and whose frequency is that of the input (of the electromotive force).
EXAMPLE 1  \textit{RLC}-Circuit

Find the current \( I(t) \) in an \textit{RLC}-circuit with \( R = 11 \) Ω (ohms), \( L = 0.1 \) H (henry), \( C = 10^{-2} \) F (farad), which is connected to a source of voltage \( E(t) = 100 \sin 400t \) (hence 63\(\frac{2}{3}\) Hz = 63\(\frac{2}{3}\) cycles/\(\text{sec},\) because \( 400 = 63\frac{2}{3} \times 2 \pi \)). Assume that current and charge are zero when \( t = 0 \).

\textbf{Solution. Step 1. General solution of the homogeneous ODE.} Substituting \( R, L, C, \) and the derivative \( E'(t) \) into (1), we obtain

\[
0.1 I'' + 11 I' + 100I = 100 \cdot 400 \cos 400t.
\]

Hence the homogeneous ODE is \( 0.1 I'' + 11 I' + 100I = 0 \). Its characteristic equation is

\[
0.1 \lambda^2 + 11 \lambda + 100 = 0.
\]
The roots are $\lambda_1 = -10$ and $\lambda_2 = -100$. The corresponding general solution of the homogeneous ODE is

$$I_h(t) = c_1 e^{-10t} + c_2 e^{-100t}.$$ 

**Step 2. Particular solution $I_p$ of (1).** We calculate the reactance $S = 40 - 1/4 = 39.75$ and the steady-state current

$$I_p(t) = a \cos 400t + b \sin 400t$$

with coefficients obtained from (4)

$$a = \frac{-100 \cdot 39.75}{11^2 + 39.75^2} = -2.3368, \quad b = \frac{100 \cdot 11}{11^2 + 39.75^2} = 0.6467.$$
Hence in our present case, a general solution of the nonhomogeneous ODE (1) is

\[
I(t) = c_1 e^{-10t} + c_2 e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.
\]

Step 3. Particular solution satisfying the initial conditions. How to use \( Q(0) = 0 \)? We finally determine \( c_1 \) and \( c_2 \) from the initial conditions \( I(0) = 0 \) and \( Q(0) = 0 \). From the first condition and (6) we have

\[
I(0) = c_1 + c_2 - 2.3368 = 0,
\]

hence

\[
c_2 = 2.3368 - c_1.
\]
Furthermore, using (1′) with $t = 0$ and noting that the integral equals $Q(t)$ (see the formula before (1′)), we obtain

$$LI'(0) + R \cdot 0 + \frac{1}{C} \cdot 0 = 0,$$

hence $$I'(0) = 0.$$ Differentiating (6) and setting $t = 0$, we thus obtain

$$I'(0) = -10c_1 - 100c_2 + 0 + 0.6467 \cdot 400 = 0,$$

hence $$-10c_1 = 100(2.3368 - c_1) - 258.68.$$
The solution of this and (7) is $c_1 = -0.2776$, $c_2 = 2.6144$. Hence the answer is

$$I(t) = -0.2776e^{-10t} + 2.6144e^{-100t} - 2.3368 \cos 400t + 0.6467 \sin 400t.$$  

Figure 62 shows $I(t)$ as well as $I_p(t)$, which practically coincide, except for a very short time near $t = 0$ because the exponential terms go to zero very rapidly. Thus after a very short time the current will practically execute harmonic oscillations of the input frequency $63(2/3)$ Hz = $63(2/3)$ cycles/sec. Its maximum amplitude and phase lag can be seen from (5), which here takes the form

$$I_p(t) = 2.4246 \sin (400t - 1.3008).$$
Fig. 62. Transient and steady-state currents in Example 1
Entirely different physical or other systems may have the same mathematical model. For instance, we have seen this from the various applications of the ODE $y' = ky$ in Chap. 1. Another impressive demonstration of this unifying power of mathematics is given by the ODE (1) for an electric RLC-circuit and the ODE (2) in the last section for a mass–spring system. Both equations

$$LI'' + RI' + \frac{1}{C} I = E_0 \omega \cos \omega t$$

and

$$my'' + cy' + ky = F_0 \cos \omega t$$
are of the same form. Table 2.2 shows the analogy between the various quantities involved. The inductance $L$ corresponds to the mass $m$ and, indeed, an inductor opposes a change in current, having an "inertia effect" similar to that of a mass. The resistance $R$ corresponds to the damping constant $c$, and a resistor causes loss of energy, just as a damping dashpot does. And so on.
This analogy is *strictly quantitative* in the sense that to a given mechanical system we can construct an electric circuit whose current will give the exact values of the displacement in the mechanical system when suitable scale factors are introduced.

The *practical importance* of this analogy is almost obvious. The analogy may be used for constructing an “electrical model” of a given mechanical model, resulting in substantial savings of time and money because electric circuits are easy to assemble, and electric quantities can be measured much more quickly and accurately than mechanical ones.
### Chapter 2 Second-Order Linear ODEs

#### Table 2.2 Analogy of Electrical and Mechanical Quantities

<table>
<thead>
<tr>
<th>Electrical System</th>
<th>Mechanical System</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inductance $L$</td>
<td>Mass $m$</td>
</tr>
<tr>
<td>Resistance $R$</td>
<td>Damping constant $c$</td>
</tr>
<tr>
<td>Reciprocal 1/$C$ of capacitance</td>
<td>Spring modulus $k$</td>
</tr>
<tr>
<td>Derivative $E_0 \omega \cos \omega t$ of</td>
<td>Driving force $F_0 \cos \omega t$</td>
</tr>
<tr>
<td>electromotive force</td>
<td>Displacement $y(t)$</td>
</tr>
<tr>
<td>Current $I(t)$</td>
<td></td>
</tr>
</tbody>
</table>
2.10 Solution by Variation of Parameters

We continue our discussion of nonhomogeneous linear ODEs

\[ y'' + p(x)y' + q(x)y = r(x). \]  

Lagrange’s method gives a particular solution \( y_p \) of (1) on \( I \) in the form

\[ y_p(x) = -y_1 \int \frac{y_2 r}{W} \, dx + y_2 \int \frac{y_1 r}{W} \, dx \]
where \( y_1, y_2 \) form a basis of solutions of the corresponding homogeneous ODE

\[
y'' + p(x)y' + q(x)y = 0
\]

on \( I \), and \( W \) is the Wronskian of \( y_1, y_2 \),

\[
W = y_1y'_2 - y_2y'_1 \quad \text{(see Sec. 2.6).}
\]
**EXAMPLE 1** Method of Variation of Parameters

Solve the nonhomogeneous ODE

\[ y'' + y = \sec x = \frac{1}{\cos x}. \]

**Solution.** A basis of solutions of the homogeneous ODE on any interval is \( y_1 = \cos x, \ y_2 = \sin x \). This gives the Wronskian

\[ W(y_1, y_2) = \cos x \cos x - \sin x (-\sin x) = 1. \]

From (2), choosing zero constants of integration, we get the particular solution of the given ODE

\[ y_p = -\cos x \int \sin x \sec x \, dx + \sin x \int \cos x \sec x \, dx \]
\[ = \cos x \ln |\cos x| + x \sin x. \]
Figure 69 shows $y_p$ and its first term, which is small, so that $x \sin x$ essentially determines the shape of the curve of $y_p$. (Recall from Sec. 2.8 that we have seen $x \sin x$ in connection with resonance, except for notation.) From $y_p$ and the general solution $y_h = c_1y_1 + c_2y_2$ of the homogeneous ODE we obtain the answer

$$y = y_h + y_p = (c_1 + \ln |\cos x|) \cos x + (c_2 + x) \sin x.$$ 

Had we included integration constants $-c_1, c_2$ in (2), then (2) would have given the additional $c_1 \cos x + c_2 \sin x = c_1y_1 + c_2y_2$, that is, a general solution of the given ODE directly from (2). This will always be the case.
Fig. 69. Particular solution $y_p$ and its first term in Example 1
Idea of the Method. Derivation of (2)

The idea is to start from a general solution

\[ y_h(x) = c_1 y_1(x) + c_2 y_2(x) \]

of the homogeneous ODE (3) on an open interval \( I \) and to replace the constants ("the parameters") \( c_1 \) and \( c_2 \) by functions \( u(x) \) and \( v(x) \); this suggests the name of the method. We shall determine \( u \) and \( v \) so that the resulting function

\[ y_p(x) = u(x)y_1(x) + v(x)y_2(x) \]
is a particular solution of the nonhomogeneous ODE (1). Note that $y_h$ exists by Theorem 3 in Sec. 2.6 because of the continuity of $p$ and $q$ on $I$. (The continuity of $r$ will be used later.)

We determine $u$ and $v$ by substituting (5) and its derivatives into (1). Differentiating (5), we obtain

$$y'_p = u'y_1 + uy'_1 + v'y_2 + vy'_2.$$
Now $y_p$ must satisfy (1). This is one condition for two functions $u$ and $v$. It seems plausible that we may impose a second condition. Indeed, our calculation will show that we can determine $u$ and $v$ such that $y_p$ satisfies (1) and $u$ and $v$ satisfy as a second condition the equation

$$u'y_1 + v'y_2 = 0.$$  

This reduces the first derivative $y'_p$ to the simpler form

$$y'_p = uy'_1 + vy'_2.$$
Differentiating (7), we obtain

\begin{equation}
  y''_p = u'y'_1 + uy''_1 + v'y'_2 + vy''_2.
\end{equation}

We now substitute \( y_p \) and its derivatives according to (5), (7), (8) into (1). Collecting terms in \( u \) and terms in \( v \), we obtain

\[
u(y''_1 + py'_1 + qy_1) + v(y''_2 + py'_2 + qy_2) + u'y'_1 + v'y'_2 = r.
\]
Since $y_1$ and $y_2$ are solutions of the homogeneous ODE (3), this reduces to

(9a) \[ u'y_1' + v'y_2' = r. \]

Equation (6) is

(9b) \[ u'y_1 + v'y_2 = 0. \]
This is a linear system of two algebraic equations for the unknown functions $u'$ and $v'$. We can solve it by elimination as follows (or by Cramer’s rule in Sec. 7.6). To eliminate $v'$, we multiply (9a) by $-y_2$ and (9b) by $y'_2$ and add, obtaining

$$u' \left( y_1 y'_2 - y_2 y'_1 \right) = -y_2 r, \quad \text{thus} \quad u'W = -y_2 r.$$ 

Here, $W$ is the Wronskian (4) of $y_1$, $y_2$. To eliminate $u'$ we multiply (9a) by $y_1$, and (9b) by $-y'_1$ and add, obtaining

$$v' \left( y_1 y'_2 - y_2 y'_1 \right) = y_1 r, \quad \text{thus} \quad v'W = y_1 r.$$
Since $y_1, y_2$ form a basis, we have $W \neq 0$ (by Theorem 2 in Sec. 2.6) and can divide by $W$,

\begin{align}
  u' &= -\frac{y_2 r}{W}, \\
  v' &= \frac{y_1 r}{W}.
\end{align}  \tag{10}

By integration,

\begin{align}
  u &= -\int \frac{y_2 r}{W} \, dx, \\
  v &= \int \frac{y_1 r}{W} \, dx.
\end{align}

These integrals exist because $r(x)$ is continuous. Inserting them into (5) gives (2) and completes the derivation.
SUMMARY OF CHAPTER 2

Second-order linear ODEs are particularly important in applications, for instance, in mechanics (Secs. 2.4, 2.8) and electrical engineering (Sec. 2.9). A second-order ODE is called linear if it can be written

$$y'' + p(x)y' + q(x)y = r(x)$$  
(Sec. 2.1).

(If the first term is, say, $f(x)y''$, divide by $f(x)$ to get the “standard form” (1) with $y''$ as the first term.) Equation (1) is called homogeneous if $r(x)$ is zero for all $x$ considered, usually in some open interval; this is written $r(x) \equiv 0$. Then

$$y'' + p(x)y' + q(x)y = 0.$$
Equation (1) is called **nonhomogeneous** if \( r(x) \neq 0 \) (meaning \( r(x) \) is not zero for some \( x \) considered).

For the homogeneous ODE (2) we have the important **superposition principle** (Sec. 2.1) that a linear combination \( y = ky_1 + ly_2 \) of two solutions \( y_1, y_2 \) is again a solution.
Two *linearly independent* solutions $y_1, y_2$ of (2) on an open interval $I$ form a **basis** (or **fundamental system**) of solutions on $I$, and $y = c_1 y_1 + c_2 y_2$ with arbitrary constants $c_1, c_2$ is a **general solution** of (2) on $I$. From it we obtain a **particular solution** if we specify numeric values (numbers) for $c_1$ and $c_2$, usually by prescribing two **initial conditions**

(3) \[ y(x_0) = K_0, \quad y'(x_0) = K_1 \]

($x_0, K_0, K_1$ given numbers; Sec. 2.1).

(2) and (3) together form an **initial value problem.** Similarly for (1) and (3).
For a nonhomogeneous ODE (1) a general solution is of the form

\[(4) \quad y = y_h + y_p \quad \text{(Sec. 2.7)} \]

Here \(y_h\) is a general solution of (2) and \(y_p\) is a particular solution of (1). Such a \(y_p\) can be determined by a general method (variation of parameters, Sec. 2.10) or in many practical cases by the method of undetermined coefficients. The latter applies when (1) has constant coefficients \(p\) and \(q\), and \(r(x)\) is a power of \(x\), sine, cosine, etc. (Sec. 2.7). Then we write (1) as

\[(5) \quad y'' + ay' + by = r(x) \quad \text{(Sec. 2.7)} \]
The corresponding homogeneous ODE $y' + ay' + by = 0$ has solutions $y = e^{\lambda x}$, where $\lambda$ is a root of

$$\lambda^2 + a\lambda + b = 0.$$  

Hence there are three cases (Sec. 2.2):

<table>
<thead>
<tr>
<th>Case</th>
<th>Type of Roots</th>
<th>General Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>Distinct real $\lambda_1, \lambda_2$</td>
<td>$y = c_1e^{\lambda_1x} + c_2e^{\lambda_2x}$</td>
</tr>
<tr>
<td>II</td>
<td>Double $-\frac{1}{2}a$</td>
<td>$y = (c_1 + c_2x)e^{-ax/2}$</td>
</tr>
<tr>
<td>III</td>
<td>Complex $-\frac{1}{2}a \pm i\omega^*$</td>
<td>$y = e^{-ax/2}(A \cos \omega^*x + B \sin \omega^*x)$</td>
</tr>
</tbody>
</table>

Important applications of (5) in mechanical and electrical engineering in connection with 

**vibrations** and **resonance** are discussed in Secs. 2.4, 2.7, and 2.8.
Another large class of ODEs solvable “algebraically” consists of the **Euler–Cauchy equations**

\[ x^2 y'' + axy' + by = 0 \]  
(Sec. 2.5).

These have solutions of the form \( y = x^m \), where \( m \) is a solution of the auxiliary equation

\[ m^2 + (a - 1)m + b = 0. \]

**Existence and uniqueness** of solutions of (1) and (2) is discussed in Secs. 2.6 and 2.7, and **reduction of order** in Sec. 2.1.