Chapter 3 High-Order Linear ODEs

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3.1 Homogeneous Linear ODEs

Recall from Sec. 1.1 that an ODE is of \textit{nth order} if the \textit{nth} derivative \( y^{(n)} = \frac{d^n y}{dx^n} \) of the unknown function \( y(x) \) is the highest occurring derivative. Thus the ODE is of the form

\[
F(x, y, y', \cdots, y^{(n)}) = 0 \quad \left(y^{(n)} = \frac{d^n y}{dx^n}\right)
\]

where lower order derivatives and \( y \) itself may or may not occur. Such an ODE is called \textit{linear} if it can be written

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x).
\]
(For $n = 2$ this is (1) in Sec. 2.1 with $p_1 = p$ and $p_0 = q$). The coefficients $p_0, \ldots, p_{n-1}$ and the function $r$ on the right are any given functions of $x$, and $y$ is unknown. $y^{(n)}$ has coefficient 1. This is practical. We call this the **standard form**. (If you have $p_n(x)y^{(n)}$, divide by $p_n(x)$ to get this form.) An $n$th-order ODE that cannot be written in the form (1) is called **nonlinear**.
If \( r(x) \) is identically zero, \( r(x) \equiv 0 \) (zero for all \( x \) considered, usually in some open interval \( I \)), then (1) becomes

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0
\]

and is called **homogeneous**. If \( r(x) \) is not identically zero, then the ODE is called **nonhomogeneous**. This is as in Sec. 2.1.
A **solution** of an $n$th-order (linear or nonlinear) ODE on some open interval $I$ is a function $y = h(x)$ that is defined and $n$ times differentiable on $I$ and is such that the ODE becomes an identity if we replace the unknown function $y$ and its derivatives by $h$ and its corresponding derivatives.
THEOREM 1

Fundamental Theorem for the Homogeneous Linear ODE (2)

For a homogeneous linear ODE (2), sums and constant multiples of solutions on some open interval I are again solutions on I. (This does not hold for a nonhomogeneous or nonlinear ODE!)
DEFINITION

General Solution, Basis, Particular Solution

A general solution of (2) on an open interval I is a solution of (2) on I of the form

\[ y(x) = c_1y_1(x) + \cdots + c_ny_n(x) \quad (c_1, \cdots, c_n \text{ arbitrary}) \]

where \( y_1, \cdots, y_n \) is a basis (or fundamental system) of solutions of (2) on I; that is, these solutions are linearly independent on I, as defined below.

A particular solution of (2) on I is obtained if we assign specific values to the \( n \) constants \( c_1, \cdots, c_n \) in (3).
Linear Independence and Dependence

$n$ functions $y_1(x)$, $\cdots$, $y_n(x)$ are called linearly independent on some interval $I$ where they are defined if the equation

\[ k_1y_1(x) + \cdots + k_ny_n(x) = 0 \quad \text{on } I \]

implies that all $k_1, \cdots, k_n$ are zero. These functions are called linearly dependent on $I$ if this equation also holds on $I$ for some $k_1, \cdots, k_n$ not all zero.
If and only if \( y_1, \ldots, y_n \) are linearly dependent on \( I \), we can express (at least) one of these functions on \( I \) as a "linear combination" of the other \( n - 1 \) functions, that is, as a sum of those functions, each multiplied by a constant (zero or not). This motivates the term “linearly dependent.” For instance, if (4) holds with \( k_1 \neq 0 \), we can divide by \( k_1 \) and express \( y_1 \) as the linear combination

\[
y_1 = -\frac{1}{k_1}(k_2y_2 + \cdots + k_ny_n)
\]

Note that when \( n = 2 \), these concepts reduce to those defined in Sec. 2.1.
EXAMPLE 1  Linear Dependence

Show that the functions $y_1 = x^2$, $y_2 = 5x$, $y_3 = 2x$ are linearly dependent on any interval.

Solution. $y_2 = 0y_1 + 2.5y_3$. This proves linear dependence on any interval.
EXAMPLE 2 Linear Independence

Show that $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$ are linearly independent on any interval, for instance, on $-1 \leq x \leq 2$.

Solution. Equation (4) is $k_1 x + k_2 x^2 + k_3 x^3 = 0$. Taking (a) $x = -1$, (b) $x = 1$, (c) $x = 2$, we get

(a) $-k_1 + k_2 - k_3 = 0$,  \hspace{1cm} (b) $k_1 + k_2 + k_3 = 0$,  
(c) $2k_1 + 4k_2 + 8k_3 = 0$.

$k_2 = 0$ from (a) + (b). Then $k_3 = 0$ from (c) – 2(b). Then $k_1 = 0$ from (b). This proves linear independence.

A better method for testing linear independence of solutions of ODEs will soon be explained.
EXAMPLE 3 General Solution. Basis

Solve the fourth-order ODE

\[ y^{iv} - 5y'' + 4y = 0 \quad (\text{where } y^{iv} = d^4y/dx^4). \]

Solution. As in Sec. 2.2 we try and substitute \( y = e^{\lambda x} \). Omitting the common factor \( e^{\lambda x} \), we obtain the characteristic equation

\[ \lambda^4 - 5\lambda^2 + 4 = 0. \]
This is a quadratic equation in $\mu = \lambda^2$, namely,

\[ \mu^2 - 5\mu + 4 = (\mu - 1)(\mu - 4) = 0. \]

The roots are $\mu = 1$ and $4$. Hence $\lambda = -2, -1, 1, 2$. This gives four solutions. A general solution on any interval is

\[ y = c_1 e^{-2x} + c_2 e^x + c_3 e^x + c_4 e^{2x} \]

provided those four solutions are linearly independent. This is true but will be shown later.
Initial Value Problem. Existence and Uniqueness

An initial value problem for the ODE (2) consists of (2) and \( n \) initial conditions

\[
y(x_0) = K_0, \quad y'(x_0) = K_1, \quad \ldots, \quad y^{(n-1)}(x_0) = K_{n-1}
\]

with given \( x_0 \) in the open interval \( I \) considered, and given \( K_0, \ldots, K_{n-1} \).

In extension of the existence and uniqueness theorem in Sec. 2.6 we now have the following.
Existence and Uniqueness Theorem for Initial Value Problems

If the coefficients $p_0(x)$, $\ldots$, $p_{n-1}(x)$ of (2) are continuous on some open interval $I$ and $x_0$ is in $I$, then the initial value problem (2), (5) has a unique solution $y(x)$ on $I$. 
EXAMPLE 4 Initial Value Problem for a Third-Order Euler–Cauchy Equation

Solve the following initial value problem on any open interval \( I \) on the positive \( x \)-axis containing \( x = 1 \).

\[
x^3y''' - 3x^2y'' + 6xy' - 6y = 0,
\]

\[
y(1) = 2, \ y'(1) = 1, \ y''(1) = -4.
\]

Solution. Step 1. General solution. As in Sec. 2.5 we try \( y = x^m \). By differentiation and substitution,

\[
m(m - 1)(m - 2)x^m - 3m(m - 1)x^m + 6mx^m - 6x^m = 0.
\]
Dropping $x^m$ and ordering gives $m^3 - 6m^2 + 11m - 6 = 0$. If we can guess the root $m = 1$, we can divide by $m - 1$ and find the other roots 2 and 3, thus obtaining the solutions $x, x^2, x^3$, which are linearly independent on $I$ (see Example 2). [In general one shall need a root-finding method, such as Newton’s (Sec. 19.2), also available in a CAS (Computer Algebra System).] Hence a general solution is

$$y = c_1x + c_2x^2 + c_3x^3$$

valid on any interval $I$, even when it includes $x = 0$ where the coefficients of the ODE divided by $x^3$ (to have the standard form) are not continuous.
Step 2. Particular solution. The derivatives are \( y' = c_1 + 2c_2 x + 3c_3 x^2 \) and \( y'' = 2c_2 + 6c_3 x \). From this and \( y \) and the initial conditions we get by setting \( x = 1 \)

(a) \( y(1) = c_1 + c_2 + c_3 = 2 \)

(b) \( y'(1) = c_1 + 2c_2 + 3c_3 = 1 \)

(c) \( y''(1) = 2c_2 + 6c_3 = -4 \).

This is solved by Cramer’s rule (Sec. 7.6), or by elimination, which is simple, as follows. (b) – (a) gives (d) \( c_2 + 2c_3 = -1 \). Then (c) – 2(d) gives \( c_3 = -1 \). Then (c) gives \( c_2 = 1 \). Finally \( c_1 = 2 \) from (a).

Answer: \( y = 2x + x^2 - x^3 \).
Linear Independence of Solutions. Wronskian

Linear independence of solutions is crucial for obtaining general solutions. Although it can often be seen by inspection, it would be good to have a criterion for it. Now Theorem 2 in Sec. 2.6 extends from order $n = 2$ to any $n$. This extended criterion uses the Wronskian $W$ of $n$ solutions $y_1, \ldots, y_n$ defined as the $n$th order determinant
Note that $W$ depends on $x$ since $y_1, \cdots, y_n$ does. The criterion states that these solutions form a basis if and only if $W$ is not zero; more precisely.
THEOREM 3

Linear Dependence and Independence of Solutions

Let the ODE (2) have continuous coefficients \( p_0(x), \ldots, p_{n-1}(x) \) on an open interval \( I \). Then \( n \) solutions \( y_1, \ldots, y_n \) of (2) on \( I \) are linearly dependent on \( I \) if and only if their Wronskian is zero for some \( x = x_0 \) in \( I \). Furthermore, if \( W \) is zero for \( x = x_0 \), then \( W \) is identically zero on \( I \). Hence if there is an \( x_1 \) in \( I \) at which \( W \) is not zero, then \( y_1, \ldots, y_n \) are linearly independent on \( I \), so that they form a basis of solutions of (2) on \( I \).
EXAMPLE 5  Basis, Wronskian

We can now prove that in Example 3 we do have a basis. In evaluating $W$, pull out the exponential functions columnwise. In the result, subtract Column 1 from Columns 2, 3, 4 (without changing Column 1). Then expand by Row 1. In the resulting third-order determinant, subtract Column 1 from Column 2 and expand the result by Row 2:

\[
W = \begin{vmatrix}
    e^{-2x} & e^{-x} & e^x & e^{2x} \\
    -2e^{-2x} & -e^{-x} & e^x & 2e^{2x} \\
    4e^{-2x} & e^{-x} & e^x & 4e^{2x} \\
    -8e^{-2x} & -e^{-x} & e^x & 8e^{2x}
\end{vmatrix}
= \begin{vmatrix}
    1 & 1 & 1 & 1 \\
    -2 & -1 & 1 & 2 \\
    4 & 1 & 1 & 4 \\
    -8 & -1 & 1 & 8
\end{vmatrix}
= \begin{vmatrix}
    1 & 3 & 4 \\
    -3 & -3 & 0 \\
    7 & 9 & 16
\end{vmatrix}
= 72.
\]
A General Solution of (2) Includes All Solutions

THEOREM 4

Existence of a General Solution

If the coefficients $p_0(x)$, $\cdots$, $p_{n-1}(x)$ of (2) are continuous on some open interval $I$, then (2) has a general solution on $I$. 
THEOREM 5

General Solution Includes All Solutions

If the ODE (2) has continuous coefficients $p_0(x), \ldots, p_{n-1}(x)$ on some open interval $I$, then every solution $y = Y(x)$ of (2) on $I$ is of the form

$$Y(x) = C_1y_1(x) + \cdots + C_ny_n(x)$$

where $y_1, \ldots, y_n$ is a basis of solutions of (2) on $I$ and $C_1, \ldots, C_n$ are suitable constants.
3.2 Homogeneous Linear ODEs with Constant Coefficients

In this section we consider \( n \)th-order homogeneous linear ODEs with constant coefficients, which we write in the form

\[
y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0
\]

where \( y^{(n)} = \frac{d^n y}{dx^n}, \) etc. We shall see that this extends the case \( n = 2 \) discussed in Sec. 2.2. Substituting \( y = e^{\lambda x} \) (as in Sec. 2.2), we obtain the characteristic equation

\[
\lambda^n + a_{n-1} \lambda^{(n-1)} + \cdots + a_1 \lambda + a_0 = 0
\]

of (1). If \( \lambda \) is a root of (2), then \( y = e^{\lambda x} \) is a solution of (1).
Distinct Real Roots

If all the $n$ roots $\lambda_1, \cdots, \lambda_n$ of (2) are real and different, then the $n$ solutions

\begin{align*}
y_1 &= e^{\lambda_1 x}, & \cdots & & y_n &= e^{\lambda_n x}
\end{align*}

constitute a basis for all $x$. The corresponding general solution of (1) is

\begin{align*}
y &= c_1 e^{\lambda_1 x} + \cdots + c_n e^{\lambda_n x}.
\end{align*}

Indeed, the solutions in (3) are linearly independent, as we shall see after the example.
EXAMPLE 1 Distinct Real Roots

Solve the ODE \( y'''' - 2y''' - y'' + 2y = 0 \).

Solution. The characteristic equation is \( \lambda^3 - 2\lambda^2 - \lambda + 2 = 0 \). It has the roots \(-1, 1, 2\); if you find one of them by inspection, you can obtain the other two roots by solving a quadratic equation (explain!). The corresponding general solution (4) is \( y = c_1 e^{-x} + c_2 e^x + c_3 e^{2x} \).
THEOREM 1

Basis

Solutions \[ y_1 = e^{\lambda_1 x}, \ldots, y_n = e^{\lambda_n x} \] of (1) (with any real or complex \( \lambda_j \)'s) form a basis of solutions of (1) on any open interval if and only if all \( n \) roots of (2) are different.
Linear Independence

Any number of solutions of (1) of the form $e^{\lambda x}$ are linearly independent on an open interval $I$ if and only if the corresponding $\lambda$ are all different.
Simple Complex Roots

If complex roots occur, they must occur in conjugate pairs since the coefficients of (1) are real. Thus, if \( \lambda = \gamma + i\omega \) is a simple root of (2), so is the conjugate \( \lambda = \gamma - i\omega \), and two corresponding linearly independent solutions are (as in Sec. 2.2, except for notation)

\[
y_1 = e^{\gamma x} \cos(\omega x), \quad y_2 = e^{\gamma x} \sin(\omega x).
\]
EXAMPLE 2 Simple Complex Roots.

Initial Value Problem

Solve the initial value problem

\[ y''' - y'' + 100y' - 100y = 0, \]

\[ y(0) = 4, \quad y'(0) = 11, \quad y''(0) = -299. \]

Solution. The characteristic equation is \( \lambda^3 - \lambda^2 + 100 \lambda - 100 = 0 \). It has the root 1, as can perhaps be seen by inspection. Then division by \( \lambda - 1 \) shows that the other roots are \( \pm 10i \). Hence a general solution and its derivatives (obtained by differentiation) are
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\[ y = c_1 e^x + A \cos 10x + B \sin 10x, \]
\[ y = c_1 e^x - 10A \sin 10x + 10B \cos 10x, \]
\[ y = c_1 e^x - 100A \cos 10x - 100B \sin 10x. \]

From this and the initial conditions we obtain by setting \( x = 0 \)

(a) \( c_1 + A = 4, \)
(b) \( c_1 + 10B = 11, \)
(c) \( c_1 - 100A = -299. \)
We solve this system for the unknowns $A$, $B$, $c_1$. Equation (a) minus Equation (c) gives $101A = 303$, $A = 3$. Then $c_1 = 1$ from (a) and $B = 1$ from (b). The solution is (Fig. 72)

$$y = e^x + 3 \cos 10x + \sin 10x.$$  

This gives the solution curve, which oscillates about $e^x$ (dashed in Fig. 72).
Fig. 72. Solution in Example 2
Multiple Real Roots

If a real double root occurs, say, $\lambda_1 = \lambda_2$, then $y_1 = y_2$ in (3), and we take $y_1$ and $xy_1$ as corresponding linearly independent solutions. This is as in Sec. 2.2.

More generally, if $\lambda$ is a real root of order $m$, then $m$ corresponding linearly independent solutions are

$$e^{\lambda x}, xe^\lambda, x^2 e^{\lambda x}, \ldots, x^{m-1} e^{\lambda x}.$$  

We derive these solutions after the next example and indicate how to prove their linear independence.
EXAMPLE 3  Real Double and Triple Roots

Solve the ODE $y^v - 3y^iv + 3y - y = 0$.

Solution. The characteristic equation $\lambda^5 - 3\lambda^4 + 3\lambda^3 - \lambda^2 = 0$ has the roots $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 = \lambda_4 = \lambda_5 = 1$, and the answer is

$$y = c_1 + c_2x + (c_3 + c_4x + c_5x^2)e^x.$$
Multiple Complex Roots

In this case, real solutions are obtained as for complex simple roots above. Consequently, if \( \lambda = \gamma + i\omega \) is a complex double root, so is the conjugate \( \overline{\lambda} = \gamma - i\omega \). Corresponding linearly independent solutions are

\[
\begin{align*}
(11) \quad & e^{\gamma x} \cos(\omega x), \quad e^{\gamma x} \sin(\omega x), \\
& xe^{\gamma x} \cos(\omega x), \quad xe^{\gamma x} \sin(\omega x).
\end{align*}
\]
The first two of these result from $e^{\lambda x}$ and $e^{\bar{\lambda} x}$ as before, and the second two from $xe^{\lambda x}$ and $xe^{\bar{\lambda} x}$ in the same fashion. Obviously, the corresponding general solution is

\[ y = e^{\lambda x}[(A_1 + A_2 x) \cos \omega x + (B_1 + B_2 x) \sin \omega x]. \]

For complex triple roots (which hardly ever occur in applications), one would obtain two more solutions $x^2 e^{\lambda x} \cos \omega x$, $x^2 e^{\lambda x} \sin \omega x$, and so on.
3.3 Nonhomogeneous Linear ODEs

We now turn from homogeneous to nonhomogeneous linear ODEs of \( n \)th order. We write them in standard form

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)
\]

with \( y^{(n)} = \frac{d^n y}{dx^n} \) as the first term, which is practical, and \( r(x) \neq 0 \). As for second-order ODEs, a general solution of (1) on an open interval \( I \) of the \( x \)-axis is of the form

\[
y(x) = y_h(x) + y_p(x).
\]
Here $y_h(x) = c_1y_1(x) + \cdots + c_ny_n(x)$ is a general solution of the corresponding homogeneous ODE

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0$$
on $I$. Also, $y_p$ is any solution of (1) on $I$ containing no arbitrary constants. If (1) has continuous coefficients and a continuous $r(x)$ on $I$, then a general solution of (1) exists and includes all solutions. Thus (1) has no singular solutions.
An initial value problem for (1) consists of (1) and \( n \) initial conditions

\[
y(x_0) = K_0, \quad y(x_0) = K_1, \ldots, \quad y^{(n-1)}(x_0) = K_{n-1}
\]

with \( x_0 \) in \( I \). Under those continuity assumptions it has a unique solution. The ideas of proof are the same as those for \( n = 2 \) in Sec. 2.7.
Method of Undetermined Coefficients

Equation (2) shows that for solving (1) we have to determine a particular solution of (1). For a constant-coefficient equation

\[ y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = r(x) \]  

(a₀, ⋯ , a_{n-1} constant) and special r(x) as in Sec. 2.7, such a \( y_p(x) \) can be determined by the **method of undetermined coefficients**, as in Sec. 2.7, using the following rules.
(A) **Basic Rule** as in Sec. 2.7.

(B) **Modification Rule.** If a term in your choice for \( y_p(x) \) is a solution of the homogeneous equation (3), then multiply \( y_p(x) \) by \( x^k \), where \( k \) is the smallest positive integer such that no term of \( x^k y_p(x) \) is a solution of (3).

(C) **Sum Rule** as in Sec. 2.7.
EXAMPLE 1 Initial Value Problem.
Modification Rule

Solve the initial value problem

\[(6) \quad y''' + 3y'' + 3y' + y = 30e^{-x},\]
\[y(0) = 3, \quad y'(0) = -3, \quad y''(0) = -47.\]

**Solution. Step 1.** The characteristic equation is \(\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3 = 0.\) It has the triple root \(\lambda = -1.\) Hence a general solution of the homogeneous ODE is

\[y_h = c_1e^{-x} + c_2xe^{-x} + c_3x^2e^{-x}\]
\[= (c_1 + c_2x + c_3x^2)e^{-x}.\]
Step 2. If we try $y_p = Ce^{-x}$, we get $-C + 3C - 3C + C = 30$, which has no solution. Try $Cxe^{-x}$ and $Cx^2e^{-x}$. The Modification Rule calls for

$$y_p = Cx^3e^{-x}.$$

Then

$$y'_p = C(3x^2 - x^3)e^{-x},$$

$$y''_p = C(6x - 6x^2 + x^3)e^{-x},$$

$$y'''_p = C(6 - 18x + 9x^2 - x^3)e^{-x}.$$
Substitution of these expressions into (6) and omission of the common factor $e^{-x}$ gives

$$C(6 - 18x + 9x^2 - x^3) + 3C(6x - 6x^2 + x^3)$$

$$+ 3C(3x^2 - x^3) + Cx^3 = 30.$$

The linear, quadratic, and cubic terms drop out, and $6C = 30$. Hence $C = 5$. This gives $y_p = 5x^3 e^{-x}$.  

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Step 3. We now write down \( y = y_h + y_p \), the general solution of the given ODE. From it we find \( c_1 \) by the first initial condition. We insert the value, differentiate, and determine \( c_2 \) from the second initial condition, insert the value, and finally determine \( c_3 \) from \( y''(0) \) and the third initial condition:

\[
y = y_h + y_p = (c_1 + c_2 x + c_3 x^2)e^{-x} + 5x^3e^{-x}, \quad y(0) = c_1 = 3
\]

\[
y' = [-3 + c_2 + (-c_2 + 2c_3)x + (15 - c_3)x^2 - 5x^3]e^{-x}, \quad y'(0) = -3 + c_2 = -3, \quad c_2 = 0
\]

\[
y'' = [3 + 2c_3 + (30 - 4c_3)x + (-30 + c_3)x^2 + 5x^3]e^{-x}, \quad y''(0) = 3 + 2c_3 = -47, \quad c_3 = -25.
\]
Hence the answer to our problem is (Fig. 73)

\[ y = (3 - 25x^2)e^{-x} + 5x^3e^{-x}. \]

The curve of \( y \) begins at \( (0, 3) \) with a negative slope, as expected from the initial values, and approaches zero as \( x \to \infty \). The dashed curve in Fig. 73 is \( y_p \).
Fig. 73. $y$ and $y_p$ (dashed) in Example 1
Method of Variation of Parameters

The method of variation of parameters (see Sec. 2.10) also extends to arbitrary order $n$. It gives a particular solution $y_p$ for the nonhomogeneous equation (1) (in standard form with $y^{(n)}$ as the first term!) by the formula

\[
y_p(x) = \sum_{k=1}^{n} y_k(x) \int \frac{W_k(x)}{W(x)} \ r(x) \ dx
\]

\[
= y_1(x) \int \frac{W_1(x)}{W(x)} \ r(x) \ dx + \cdots + y_n(x) \int \frac{W_n(x)}{W(x)} \ r(x) \ dx
\]
on an open interval $I$ on which the coefficients of (1) and $r(x)$ are continuous. In (7) the functions $y_1, \ldots, y_n$ form a basis of the homogeneous ODE (3), with Wronskian $W$, and $W_j$ ($j = 1, \ldots, n$) is obtained from $W$ by replacing the $j$th column of $W$ by the column $[0 \ 0 \ \cdots \ 0 \ 1]^T$. Thus, when $n = 2$, this becomes identical with (2) in Sec. 2.10,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1.$$
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EXAMPLE 2 Variation of Parameters. Nonhomogeneous Euler–Cauchy Equation

Solve the nonhomogeneous Euler–Cauchy equation

\[ x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x \quad (x > 0). \]

Solution. Step 1. General solution of the homogeneous ODE. Substitution of \( y = x^m \) and the derivatives into the homogeneous ODE and deletion of the factor \( x^m \) gives

\[ m(m - 1)(m - 2) - 3m(m - 1) + 6m - 6 = 0. \]
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The roots are 1, 2, 3 and give as a basis

\[ y_1 = x, \quad y_2 = x^2, \quad y_3 = x^3. \]

Hence the corresponding general solution of the homogeneous ODE is

\[ y_h = c_1x + c_2x^2 + c_3x^3. \]

**Step 2. Determinants needed in (7).** These are

\[
W = \begin{vmatrix}
    x & x^2 & x^3 \\
    1 & 2x & 3x^2 \\
    0 & 2 & 6x
\end{vmatrix} = 2x^3
\]
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\[ W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4 \]

\[ W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix} = -2x^3 \]

\[ W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix} = x^2. \]
Step 3. Integration. In (7) we also need the right side \( r(x) \) of our ODE in standard form, obtained by division of the given equation by the coefficient \( x^3 \) of \( y''' \); thus, \( r(x) = (x^4 \ln x)/x^3 = x \ln x \). In (7) we have the simple quotients \( W_1/W = x/2, \ W_2/W = -1, \ W_3/W = 1/(2x) \). Hence (7) becomes

\[
y_p = x \int \frac{x}{2} x \ln x \, dx - x^2 \int x \ln x \, dx + x^3 \int \frac{1}{2x} x \ln x \, dx
\]

\[
= \frac{x}{2} \left( \frac{x^3}{3} \ln x - \frac{x^3}{9} \right) - x^2 \left( \frac{x^2}{2} \ln x - \frac{x^2}{4} \right) + \frac{x^3}{2} (x \ln x - x).
\]
Simplification gives $y_p = (1/6)x^4 \ln x - 11/6$. Hence the answer is

$$y = y_h + y_p = c_1 x + c_2 x^2 + c_3 x^3 + \frac{1}{6} x^4 (\ln x - \frac{11}{6}).$$

Figure 74 shows $y_p$. Can you explain the shape of this curve? Its behavior near $x = 0$? The occurrence of a minimum? Its rapid increase? Why would the method of undetermined coefficients not have given the solution?
Fig. 74. Particular solution $y_p$ of the nonhomogeneous Euler–Cauchy equation in Example 2
Application: Elastic Beams

**EXAMPLE 3** Bending of an Elastic Beam under a Load

We consider a beam $B$ of length $L$ and constant (e.g., rectangular) cross section and homogeneous elastic material (e.g., steel). We assume that under its own weight the beam is bent so little that it is practically straight. If we apply a load to $B$ in a vertical plane through the axis of symmetry (the $x$-axis in Fig. 75), $B$ is bent. Its axis is curved into the so-called **elastic curve** $C$ (or **deflection curve**). It is shown in elasticity theory that the bending moment $M(x)$ is proportional to the curvature $k(x)$ of $C$. We assume the bending to be small, so that the
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deflection $y(x)$ and its derivative $y(x)$ (determining the tangent direction of $C$) are small. Then, by calculus, $k = y''/(1 + y'^2)^{3/2} \approx y''$. Hence

$$M(x) = E\!y''(x).$$

$EI$ is the constant of proportionality. $E$ is Young’s modulus of elasticity of the material of the beam. $I$ is the moment of inertia of the cross section about the (horizontal) $z$-axis in Fig. 75.

Elasticity theory shows further that $M''(x) = f(x)$, where $f(x)$ is the load per unit length. Together,

$$E\!y^{iv} = f(x).$$ (8)
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Fig. 75. Elastic Beam

Undeformed beam

Deformed beam under uniform load (simply supported)
The practically most important supports and corresponding boundary conditions are as follows (see Fig. 76).

(A) Simply supported \( y = y'' = 0 \) at \( x = 0 \) and \( L \)

(B) Clamped at both ends \( y = y' = 0 \) at \( x = 0 \) and \( L \)

(C) Clamped at \( x = 0 \), free at \( x = L \) \( y(0) = y'(0) = 0 \), \( y''(L) = y'''(L) = 0 \).

The boundary condition \( y = 0 \) means no displacement at that point, \( y' = 0 \) means a horizontal tangent, \( y'' = 0 \) means no bending moment, and \( y''' = 0 \) means no shear force.
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Let us apply this to the uniformly loaded simply supported beam in Fig. 75. The load is \( f(x) \equiv f_0 = \text{const.} \) Then (8) is

\[
y^{iv} = k, \quad k = \frac{f_0}{EI}.
\]

This can be solved simply by calculus. Two integrations give

\[
y'' = \frac{k}{2} x^2 + c_1x + c_2.
\]

\( y''(0) = 0 \) gives \( c_2 = 0 \). Then \( y''(L) = L(1/2 \ kL + c_1) = 0 \), \( c_1 = -kL/2 \) (since \( L \neq 0 \)). Hence

\[
y'' = \frac{k}{2} (x^2 - Lx).
\]
Integrating this twice, we obtain

\[ y = \frac{k}{2} \left( \frac{1}{12} x^4 - \frac{L}{6} x^3 + c_3 x + c_4 \right) \]

with \( c_4 = 0 \) from \( y(0) = 0 \). Then

\[ y(L) = \frac{kL}{2} \left( \frac{L^3}{12} - \frac{L^3}{6} + c_3 \right) = 0, \quad c_3 = \frac{L^3}{12}. \]

Inserting the expression for \( k \), we obtain as our solution

\[ y = \frac{f_0}{24EI} (x^4 - 2Lx^3 + L^3 x). \]
Since the boundary conditions at both ends are the same, we expect the deflection $y(x)$ to be “symmetric” with respect to $L/2$, that is, $y(x) = y(L - x)$. Verify this directly or set $x = u + L/2$ and show that $y$ becomes an even function of $u$,

$$y = \frac{f_0}{24EI} \left( u^2 - \frac{1}{4} L^2 \right) \left( u^2 - \frac{5}{4} L^2 \right).$$

From this we can see that the maximum deflection in the middle at $u = 0$ ($x = L/2$) is $5f_0L^4/(16 \cdot 24EI)$. Recall that the positive direction points downward.
Fig. 76. Supports of a Beam

(A) Simply supported

(B) Clamped at both ends

(C) Clamped at the left end, free at the right end
SUMMARY OF CHAPTER 3

*Compare with the similar Summary of Chap. 2 (the case \( n = 2 \)).*

Chapter 3 extends Chap. 2 from order \( n = 2 \) to arbitrary order \( n \). An **nth-order linear ODE** is an ODE that can be written

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = r(x)
\]
with \( y^{(n)} = d^n y/dx^n \) as the first term; we again call this the **standard form**. Equation (1) is called **homogeneous** if \( r(x) \equiv 0 \) on a given open interval \( I \) considered, **nonhomogeneous** if \( r(x) \neq 0 \) on \( I \). For the homogeneous ODE

\[
y^{(n)} + p_{n-1}(x)y^{(n-1)} + \cdots + p_1(x)y' + p_0(x)y = 0
\]
the superposition principle (Sec. 3.1) holds, just as in the case $n = 2$. A basis or fundamental system of solutions of (2) on $I$ consists of $n$ linearly independent solutions $y_1, \ldots, y_n$ of (2) on $I$. A general solution of (2) on $I$ is a linear combination of these,

$$y = c_1y_1 + \cdots + c_ny_n$$

($c_1, \ldots, c_n$ arbitrary constants).
A general solution of the nonhomogeneous ODE (1) on I is of the form

(4) \[ y = y_h + y_p \] (Sec. 3.3).

Here, \( y_p \) is a particular solution of (1) and is obtained by two methods (undetermined coefficients or variation of parameters) explained in Sec. 3.3.
An initial value problem for (1) or (2) consists of one of these ODEs and \( n \) initial conditions (Secs. 3.1, 3.3)

\[
(5) \quad y(x_0) = K_0, \quad y'(x_0) = K_1, \quad \cdots, \quad y^{(n-1)}(x_0) = K_{n-1}
\]

with given \( x_0 \) in \( I \) and given \( K_0, \cdots, K_{n-1} \). If \( p_0, \cdots, p_{n-1}, r \) are continuous on \( I \), then general solutions of (1) and (2) on \( I \) exist, and initial value problems (1), (5) or (2), (5) have a unique solution.