Chapter 5 Series Solutions of ODEs. Special Functions

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Summary of Chapter 5
5.1 Power Series Method

Power Series

From calculus we recall that a power series (in powers of $x - x_0$) is an infinite series of the form

\[ \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \]
Here, $x$ is a variable. $a_0, a_1, a_2, \ldots$ are constants, called the \textbf{coefficients} of the series. $x_0$ is a constant, called the \textbf{center} of the series. In particular, if $x_0 = 0$, we obtain a \textbf{power series in powers of $x$}

\begin{equation}
\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots.
\end{equation}

We shall assume that all variables and constants are real.
Idea of the Power Series Method

For a given ODE

\[ y'' = p(x)y' + q(x)y = 0 \]

we first represent \( p(x) \) and \( q(x) \) by power series in powers of \( x \) (or of \( x - x_0 \) if solutions in powers of \( x - x_0 \) are wanted). Often \( p(x) \) and \( q(x) \) are polynomials, and then nothing needs to be done in this first step. Next we assume a solution in the form of a power series with unknown coefficients,

\[ y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots \]
and insert this series and the series obtained by termwise differentiation,

\[
\begin{align*}
\text{(a)} \quad y' &= \sum_{m=1}^{\infty} ma_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots \\
\text{(b)} \quad y'' &= \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} = 2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + \cdots
\end{align*}
\]

(4)

into the ODE. Then we collect like powers of \( x \) and equate the sum of the coefficients of each occurring power of \( x \) to zero, starting with the constant terms, then taking the terms containing \( x \), then the terms in \( x^2 \), and so on. This gives equations from which we can determine the unknown coefficients of (3) successively.
**EXAMPLE 1**

Solve the following ODE by power series. To grasp the idea, do this by hand; do not use your CAS (for which you could program the whole process).

\[ y' = 2xy. \]

**Solution.** We insert (3) and (4a) into the given ODE, obtaining

\[ a_1 + 2a_2x + 3a_3x^2 + \cdots = 2x(a_0 + a_1x + a_2x^2 + \cdots). \]
We must perform the multiplication by $2x$ on the right and can write the resulting equation conveniently as

$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + 6a_6 x^5 + \cdots$$

$$= 2a_0 x + 2a_1 x^2 + 2a_2 x^3 + 2a_3 x^4 + 2a_4 x^5 + \cdots .$$

For this equation to hold, the two coefficients of every power of $x$ on both sides must be equal, that is,

$$a_1 = 0, \ 2a_2 = 2a_0, \ 3a_3 = 2a_1,$$
$$4a_4 = 2a_2, \ 5a_5 = 2a_3, \ 6a_6 = 2a_4, \ \cdots .$$
Hence $a_3 = 0$, $a_5 = 0$, $\cdots$ and for the coefficients with even subscripts,

\[ a_2 = a_0, \quad a_4 = \frac{a_2}{2} = \frac{a_0}{2!}, \quad a_6 = \frac{a_4}{3} = \frac{a_0}{3!}, \cdots; \]

$a_0$ remains arbitrary. With these coefficients the series (3) gives the following solution, which you should confirm by the method of separating variables.

\[
y = a_0 \left( 1 + x^2 + \frac{x^4}{2!} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots \right) = a_0 e^{x^2}.
\]
More rapidly, (3) and (4) give for the ODE \( y' = 2xy \)

\[
1 \cdot a_1 x^0 + \sum_{m=2}^{\infty} m a_m x^{m-1} = 2x \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} 2a_m x^{m+1}.
\]

Now, to get the same general power on both sides, we make a “shift of index” on the left by setting \( m = s + 2 \), thus \( m - 1 = s + 1 \). Then \( a_m \) becomes \( a_{s+2} \) and \( x^{m-1} \) becomes \( x^{s+1} \). Also the summation, which started with \( m = 2 \), now starts with \( s = 0 \) because \( s = m - 2 \). On the right we simply make a change of notation \( m = s \), hence \( a_m = a_s \) and \( x^{m+1} = x^{s+1} \); also the summation now starts with \( s = 0 \). This altogether gives
Every occurring power of $x$ must have the same coefficient on both sides; hence

\[
a_1 = 0 \quad \text{and} \quad (s + 2)a_{s+2} = 2a_s \quad \text{or} \quad a_{s+2} = \frac{2}{s + 2}a_s.
\]

For $s = 0, 1, 2, \cdots$ we thus have $a_2 = (2/2)a_0$, $a_3 = (2/3)a_1 = 0$, $a_4 = (2/4)a_2$, $\cdots$ as before.
EXAMPLE 2

Solve

\[ y'' + y = 0. \]

**Solution.** By inserting (3) and (4b) into the ODE we have

\[
\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=0}^{\infty} a_m x^m = 0.
\]

To obtain the same general power on both series, we set \( m = s + 2 \) in the first series and \( m = s \) in the second, and then we take the latter to the right side. This gives

\[
\sum_{s=0}^{\infty} (s+2)(s+1)a_{s+2} x^s = - \sum_{s=0}^{\infty} a_s x^s.
\]
Each power $x^s$ must have the same coefficient on both sides. Hence $(s + 2)(s + 1)a_{s+2} = -a_s$. This gives the recursion formula

$$a_{s+2} = -\frac{a_s}{(s + 2)(s + 1)}$$

($s = 0, 1, \cdots$).

We thus obtain successively

$$a_2 = -\frac{a_0}{2 \cdot 1} = -\frac{a_0}{2!},$$
$$a_3 = -\frac{a_1}{3 \cdot 2} = -\frac{a_1}{3!}$$

$$a_4 = -\frac{a_2}{4 \cdot 3} = \frac{a_0}{4!},$$
$$a_5 = -\frac{a_3}{5 \cdot 4} = \frac{a_1}{5!}. $$
and so on. \( a_0 \) and \( a_1 \) remain arbitrary. With these coefficients the series (3) becomes

\[
y = a_0 + a_1 x - \frac{a_0}{2!} x^2 - \frac{a_1}{3!} x^3 + \frac{a_0}{4!} x^4 + \frac{a_1}{5!} x^5 + \cdots.
\]

Reordering terms (which is permissible for a power series), we can write this in the form

\[
y = a_0 \left( 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \cdots \right) + a_1 \left( x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \cdots \right)
\]

and we recognize the familiar general solution

\[
y = a_0 \cos x + a_1 \sin x.
\]
5.2 Theory of the Power Series Method

Basic Concepts

(1) \[ \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \]

As before, we assume the variable \( x \), the center \( x_0 \), and the coefficients \( a_0, a_1, \cdots \) to be real. The \textit{nth} partial sum of (1) is

(2) \[ s_n(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 \cdots a_n(x - x_0)^n \]
where $n = 0, 1, \cdots$. Clearly, if we omit the terms of $s_n$ from (1), the remaining expression is

$$R_n(x) = a_{n+1}(x - x_0)^{n+1} + a_{n+2}(x - x_0)^{n+2} \cdots. \quad (3)$$

This expression is called the **remainder** of (1) after the term $a_n(x - x_0)^n$. 
Geometrically, this means that all $s_n(x_1)$ with $n > N$ lie between $s(x_1) - \varepsilon$ and $s(x_1) + \varepsilon$. Practically, this means that in the case of convergence we can approximate the sum $s(x_1)$ of (1) at $x_1$ by $s_n(x_1)$ as accurately as we please, by taking $n$ large enough.

Fig. 102. Inequality (5)
Convergence Interval. Radius of Convergence

With respect to the convergence of the power series (1) there are three cases, the useless Case 1, the usual Case 2, and the best Case 3, as follows.

**Case 1.** The series (1) always converges at \( x = x_0 \), because for \( x = x_0 \) all its terms are zero, perhaps except for the first one, \( a_0 \). In exceptional cases \( x = x_0 \) may be the only \( x \) for which (1) converges. Such a series is of no practical interest.
Case 2. If there are further values of $x$ for which the series converges, these values form an interval, called the **convergence interval**. If this interval is finite, it has the midpoint $x_0$, so that it is of the form

\[ |x - x_0| < R \]  

![Diagram of convergence interval](image)

**Fig. 103.** Convergence interval (6) of a power series with center $x_0$
and the series (1) converges for all \( x \) such that \( |x - x_0| < R \) and diverges for all \( x \) such that \( |x - x_0| > R \). (No general statement about convergence or divergence can be made for \( x - x_0 = R \) or \( -R \).) The number \( R \) is called the **radius of convergence** of (1). \( R \) is called “radius” because for a complex power series it is the radius of a disk of convergence.) \( R \) can be obtained from either of the formulas

\[
\begin{align*}
\text{(a)} & \quad R = 1 \lim_{m \to \infty} \sqrt[m]{|a_m|} \\
\text{(b)} & \quad R = 1 \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right|
\end{align*}
\]
provided these limits exist and are not zero. [If these limits are infinite, then (1) converges only at the center $x_0$.]

**Case 3.** The convergence interval may sometimes be infinite, that is, (1) converges for all $x$. For instance, if the limit in (7a) or (7b) is zero, this case occurs. One then writes $R = \infty$, for convenience.
EXAMPLE 1  The Useless Case 1 of Convergence Only at the Center

In the case of the series

\[ \sum_{m=0}^{\infty} m! x^m = 1 + x + 2x^2 + 6x^3 + \cdots \]

we have \( a_m = m! \), and in (7b),

\[ \frac{a_{m+1}}{a_m} = \frac{(m + 1)!}{m!} = m + 1 \rightarrow \infty \quad \text{as} \quad m \rightarrow \infty. \]

Thus this series converges only at the center \( x = 0 \). Such a series is useless.
EXAMPLE 2 The Usual Case 2 of Convergence in a Finite Interval. Geometric Series

For the **geometric series** we have

\[ \frac{1}{1 - x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \cdots \quad (|x| < 1). \]

In fact, \( a_m = 1 \) for all \( m \), and from (7) we obtain \( R = 1 \), that is, the geometric series converges and represents \( 1/(1 - x) \) when \( |x| < 1 \).
EXAMPLE 3 The Best Case 3 of Convergence for All $x$

In the case of the series
\[ e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \cdots \]
we have $a_m = 1/m!$. Hence in (7b),
\[ \frac{a_{m+1}}{a_m} = \frac{1/(m+1)!}{1/m!} = \frac{1}{m+1} \rightarrow 0 \quad \text{as } m \rightarrow \infty, \]
so that the series converges for all $x$. 

Chapter 5 Series Solutions of ODEs. Special Functions
EXAMPLE 4  Hint for Some of the Problems

Find the radius of convergence of the series

\[ \sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^{3m} = 1 - \frac{x^3}{8} + \frac{x^6}{64} - \frac{x^9}{512} + \cdots. \]

Solution. This is a series in powers of \( t = x^3 \) with coefficients \( a_m = (-1)^m/8^m \), so that in (7b),

\[ \left| \frac{a_{m+1}}{a_m} \right| = \frac{8^m}{8^{m+1}} = \frac{1}{8}. \]

Thus \( R = 8 \). Hence the series converges for \( |t| = |x^3| < 8 \), that is, \( |x| < 2 \).
Operations on Power Series

Termwise Differentiation

A power series may be differentiated term by term. More precisely: if

\[ y(x) = \sum_{m=0}^{\infty} a_m(x - x_0)^m \]

converges for \( |x - x_0| < R \), where \( R > 0 \), then the series obtained by differentiating term by term also converges for those \( x \) and represents the derivative \( y' \) of \( y \) for those \( x \), that is,

\[ y'(x) = \sum_{m=1}^{\infty} ma_m(x - x_0)^{m-1} \quad (|x - x_0| < R). \]
Similarly,

\[ y''(x) = \sum_{m=2}^{\infty} m(m-1)a_m(x - x_0)^{m-2} \quad (|x - x_0| < R), \text{ etc.} \]

**Termwise Addition**

*Two power series may be added term by term.* More precisely: if the series

(8) \[ \sum_{m=0}^{\infty} a_m(x - x_0)^m \quad \text{and} \quad \sum_{m=0}^{\infty} b_m(x - x_0)^m \]
have positive radii of convergence and their sums are $f(x)$ and $g(x)$, then the series

$$
\sum_{m=0}^{\infty} (a_m + b_m)(x - x_0)^m
$$

converges and represents $f(x) + g(x)$ for each $x$ that lies in the interior of the convergence interval of each of the two given series.

**Termwise Multiplication**

*Two power series may be multiplied term by term.* More precisely: Suppose that the series (8) have positive radii of convergence and let $f(x)$ and $g(x)$ be their sums.
Then the series obtained by multiplying each term of the first series by each term of the second series and collecting like powers of \( x - x_0 \), that is,

\[
\sum_{m=0}^{\infty} (a_0 b_m + a_1 b_{m-1} + \cdots + a_m b_0)(x - x_0)^m
\]

\[
= a_0 b_0 + (a_0 b_1 + a_1 b_0)(x - x_0) + (a_0 b_2 + a_1 b_1 + a_2 b_0)(x - x_0)^2 + \cdots
\]

converges and represents \( f(x)g(x) \) for each \( x \) in the interior of the convergence interval of each of the two given series.
Existence of Power Series Solutions of ODEs. Real Analytic Functions

**DEFINITION**

Real Analytic Function

A real function $f(x)$ is called analytic at a point $x = x_0$ if it can be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$. 
THEOREM 1

Existence of Power Series Solutions

If $p$, $q$, and $r$ in (9) are analytic at $x = x_0$, then every solution of (9) is analytic at $x = x_0$ and can thus be represented by a power series in powers of $x - x_0$ with radius of convergence $R > 0$. Hence the same is true if $\tilde{h}$, $\tilde{p}$, $\tilde{q}$, and $\tilde{r}$ in (10) are analytic at $x = x_0$ and $\tilde{h}(x_0) \neq 0$. 
5.3 Legendre’s Equation.
Legendre Polynomials $P_n(x)$

This is Legendre’s equation

\[(1 - x^2)y'' - 2xy' + n(n + 1)y = 0\]

where $n$ is a given constant. Legendre’s equation arises in numerous problems, particularly in boundary value problems for spheres (take a quick look at Example 1 in Sec. 12.10). The parameter $n$ in (1) is a given real number. Any solution of (1) is called a Legendre function.
Dividing (1) by the coefficient $1 - x^2$ of $y''$, we see that the coefficients $-2x/(1 - x^2)$ and $n(n + 1)/(1 - x^2)$ of the new equation are analytic at $x = 0$. Hence by Theorem 1, in Sec. 5.2, Legendre’s equation has power series solutions of the form

$$y = \sum_{m=0}^{\infty} a_m x^m. \quad (2)$$

Substituting (2) and its derivatives into (1), and denoting the constant $n(n + 1)$ simply by $k$, we obtain

$$\sum_{m=2}^{\infty} m(m-1)a_{m-2}x^{m-2} - 2x \sum_{m=1}^{\infty} ma_{m-1}x^{m-1} + k \sum_{m=0}^{\infty} a_m x^m = 0.$$
By writing the first expression as two separate series we have the equation

\[
\sum_{m=2}^{\infty} m(m - 1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m - 1)a_m x^m - \sum_{m=1}^{\infty} 2ma_m x^m + \sum_{m=0}^{\infty} k a_m x^m = 0.
\]

To obtain the same general power \(x^s\) in all four series, we set \(m - 2 = s\) (thus \(m = s + 2\)) in the first series and simply write \(s\) instead of \(m\) in the other three series. This gives

\[
\sum_{s=0}^{\infty} (s + 2)(s + 1)a_{s+2} x^s - \sum_{s=2}^{\infty} s(s - 1)a_s x^s - \sum_{s=1}^{\infty} 2sa_s x^s + \sum_{s=0}^{\infty} k a_s x^s = 0.
\]
We obtain the general formula

\[ a_{s+2} = - \frac{(n-s)(n+s+1)}{(s+2)(s+1)} a_s \quad (s = 0, 1, \cdots). \]

and so on. By inserting these expressions for the coefficients into (2) we obtain

\[ y(x) = a_0 y_1(x) + a_1 y_2(x) \]
where

\[ y_1(x) = 1 - \frac{n(n + 1)}{2!} x^2 + \frac{(n - 2)n(n + 1)(n + 3)}{4!} x^4 - + \ldots \]

\[ y_2(x) = x - \frac{(n - 1)(n + 2)}{3!} x^3 + \frac{(n - 3)(n - 1)(n + 2)(n + 4)}{5!} x^5 - + \ldots \]

These series converge for \( |x| < 1 \).
Legendre Polynomials $P_n(x)$

In various applications, power series solutions of ODEs reduce to polynomials, that is, they terminate after finitely many terms. This is a great advantage and is quite common for special functions, leading to various important families of polynomials (see Refs. [GR1] or [GR10] in App. 1). For Legendre’s equation this happens when the parameter $n$ is a nonnegative integer because then the right side of (4) is zero for $s = n$, so that $a_{n+2} = 0$, $a_{n+4} = 0$, $a_{n+6} = 0$, $\ldots$. Hence if $n$ is even, $y_1(x)$ reduces to a polynomial of degree $n$. If $n$ is odd, the same is true for $y_2(x)$. These polynomials, multiplied by some constants, are called Legendre polynomials and are denoted by $P_n(x)$. The standard choice of a constant is done as follows. We choose the coefficient $a_{n}$ of the highest power $x^n$ as
(8) \[ a_n = \frac{(2n)!}{2^n(n!)^2} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{n!} \] (\text{n a positive integer})

(and \( a_n = 1 \) if \( n = 0 \)). Then we calculate the other coefficients from (4), solved for \( a_s \) in terms of \( a_{s+2} \), that is,

(9) \[ a_s = -\frac{(s+2)(s+1)}{(n-s)(n+s+1)} a_{s+2} \] (\( s \leq n-2 \)).

The choice (8) makes \( P_n(1) = 1 \) for every \( n \) (see Fig. 104); this motivates (8).
The resulting solution of Legendre’s differential equation (1) is called the **Legendre polynomial of degree** $n$ and is denoted by $P_n(x)$.

From (10) we obtain

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^m m! (n-m)! (n-2m)!} x^{n-2m}$$

$$= \frac{(2n)!}{2^n (n!)^2} x^n - \frac{(2n-2)!}{2^n 1! (n-1)! (n-2)!} x^{n-2} + \cdots$$
where $M = n/2$ or $(n - 1)/2$, whichever is an integer. The first few of these functions are (Fig. 104)

\[
\begin{align*}
P_0(x) &= 1, \\
P_1(x) &= x, \\
P_2(x) &= \frac{1}{2}(3x^2 - 1), \\
P_3(x) &= \frac{1}{2}(5x^3 - 3x), \\
P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3), \\
P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x)
\end{align*}
\]

and so on. You may now program (11) on your CAS and calculate $P_n(x)$ as needed.

The so-called **orthogonality** of the Legendre polynomials will be considered in Secs. 5.7 and 5.8.
Fig. 104. Legendre polynomials
THEOREM 1

Let $b(x)$ and $c(x)$ be any functions that are analytic at $x = 0$. Then the ODE

$$y'' + \frac{b(x)}{x} y' + \frac{c(x)}{x^2} y = 0$$

(1)

has at least one solution that can be represented in the form

$$y(x) = x^r \sum_{m=0}^{\infty} a_m x^m = x^r(a_0 + a_1 x + a_2 x^2 + \cdots)$$

(2)

where the exponent $r$ may be any (real or complex) number (and $r$ is chosen so that $a_0 \neq 0$).
THEOREM 1

Frobenius Method(2)

The ODE (1) also has a second solution (such that these two solutions are linearly independent) that may be similar to (2) (with a different \( r \) and different coefficients) or may contain a logarithmic term. (Details in Theorem 2 below.)
Regular and Singular Points

The following commonly used terms are practical. A regular point of

\[ y'' + p(x)y' + q(x)y = 0 \]

is a point \( x_0 \) at which the coefficients \( p \) and \( q \) are analytic. Then the power series method can be applied. If \( x_0 \) is not regular, it is called singular. Similarly, a regular point of the ODE

\[ \tilde{h}(x)y'' + \tilde{p}(x)y'(x) + \tilde{q}(x)y = 0 \]

is an \( x_0 \) at which \( \tilde{h}, \tilde{p}, \tilde{q} \) are analytic and \( \tilde{h}(x_0) \neq 0 \) (so what we can divide by \( \tilde{h} \) and get the previous standard form). If \( x_0 \) is not regular, it is called singular.
Indicial Equation, Indicating the Form of Solutions

We shall now explain the Frobenius method for solving (1). Multiplication of (1) by \( x^2 \) gives the more convenient form

\[
(1') \quad x^2 y'' + xb(x)y' + c(x)y = 0.
\]

We first expand \( b(x) \) and \( c(x) \) in power series,

\[
b(x) = b_0 + b_1 x + b_2 x^2 + \cdots, \\
c(x) = c_0 + c_1 x + c_2 x^2 + \cdots
\]

or we do nothing if \( b(x) \) and \( c(x) \) are polynomials. Then we differentiate (2) term by term, finding

\[
y'(x) = \sum_{m=0}^{\infty} (m + r) a_m x^{m+r-1} = x^{r-1} \left[ ra_0 + (r + 1) a_1 x + \cdots \right]
\]
By inserting all these series into (1') we readily obtain

\[ x^r[r(r-1)a_0 + \cdots] + (b_0 + b_1x + \cdots)x^r(ra_0 + \cdots) + (c_0 + c_1x + \cdots)x^r(a_0 + a_1x + \cdots) = 0. \]

(3)
We now equate the sum of the coefficients of each power $x^r, x^{r+1}, x^{r+2}, \cdots$ to zero. This yields a system of equations involving the unknown coefficients $a_m$. The equation corresponding to the power $x^r$ is

$$[r(r-1) + b_0r + c_0]a_0 = 0.$$ 

Since by assumption $a_0 \neq 0$, the expression in the brackets $[\cdots]$ must be zero. This gives

$$r(r-1) + b_0r + c_0 = 0. \quad (4)$$

This important quadratic equation is called the indicial equation of the ODE (1).
Suppose that the ODE (1) satisfies the assumptions in Theorem 1. Let \( r_1 \) and \( r_2 \) be the roots of the indicial equation (4). Then we have the following three cases.

**Case 1. Distinct Roots Not Differing by an Integer.** A basis is

\[
y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots)
\]

and

\[
y_2(x) = x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots)
\]
THEOREM 2

Frobenius Method. Basis of Solutions. Three Cases(2)

Case 2. Double Root $r_1 = r_2 = r$. A basis is

\[ y_1(x) = x^r(a_0 + a_1 x + a_2 x^2 + \cdots) \quad [r = \frac{1}{2}(1 - b_0)] \]

(of the same general form as before) and

\[ y_2(x) = y_1(x) \ln x + x^r(A_1 x + A_2 x^2 + \cdots) \quad (x > 0). \]
THEOREM 2

Frobenius Method. Basis of Solutions.  
Three Cases (3)

Case 3. Roots Differing by an Integer. A basis is

\[ y_1(x) = x^{r_1}(a_0 + a_1x + a_2x^2 + \cdots) \]  \hspace{1cm} (9)

(\textit{of the same general form as before}) and

\[ y_2(x) = ky_1(x) \ln x + x^{r_2}(A_0 + A_1x + A_2x^2 + \cdots), \]  \hspace{1cm} (10)

where the roots are so denoted that \( r_1 - r_2 > 0 \) and \( k \) may turn out to be zero.
Chapter 5 Series Solutions of ODEs. Special Functions

Typical Applications

Example 1 Euler–Cauchy Equation, Illustrating Cases 1 and 2 and Case 3 without a Logarithm

For the Euler–Cauchy equation (Sec. 2.5)

\[ x^2 y'' + b_0 x y' + c_0 y = 0 \] (\( b_0, c_0 \) constant)

substitution of \( y = x^r \) gives the auxiliary equation

\[ r(r - 1) + b_0 r + c_0 = 0, \]

which is the indicial equation [and \( y = x^r \) is a very special form of (2)!!]. For different roots \( r_1, r_2 \) we get a basis \( y_1 = x^{r_1}, y_2 = x^{r_2} \), and for a double root \( r \) we get a basis \( x^r, x^r \ln x \). Accordingly, for this simple ODE, Case 3 plays no extra role.
EXAMPLE 2 Illustration of Case 2 (Double Root)

Solve the ODE

\[ x(x - 1)y'' + (3x - 1)y' + y = 0. \]  
(This is a special hypergeometric equation, as we shall see in the problem set.)

Solution. Writing (11) in the standard form (1), we see that it satisfies the assumptions in Theorem 1. [What are \( b(x) \) and \( c(x) \) in (11)?] By inserting (2) and its derivatives (2*) into (11) we obtain
The smallest power is $x^{r-1}$, occurring in the second and the fourth series; by equating the sum of its coefficients to zero we have

$$[-r(r-1) - r]a_0 = 0, \quad \text{thus} \quad r^2 = 0.$$ 

Hence this indicial equation has the double root $r = 0$. 
**First Solution.** We insert this value \( r = 0 \) into (12) and equate the sum of the coefficients of the power \( x^s \) to zero, obtaining

\[
s(s - 1)a_s - (s + 1)sa_{s+1} + 3sa_s - (s + 1)a_{s+1} + a_s = 0
\]

thus \( a_{s+1} = a_s \). Hence \( a_0 = a_1 = a_2 = \cdots \), and by choosing \( a_0 = 1 \) we obtain the solution

\[
y_1(x) = \sum_{m=0}^{\infty} x^m = \frac{1}{1 - x}
\]

\((|x| < 1)\).
Second Solution. We get a second independent solution $y_2$ by the method of reduction of order (Sec. 2.1), substituting $y_2 = uy_1$ and its derivatives into the equation. This leads to (9), Sec. 2.1, which we shall use in this example, instead of starting reduction of order from scratch (as we shall do in the next example). In (9) of Sec. 2.1 we have $p = (3x - 1)/(x^2 - x)$, the coefficient of $y$ in (11) in standard form. By partial fractions,

$$- \int p \, dx = - \int \frac{3x - 1}{x(x - 1)} \, dx = - \int \left( \frac{2}{x - 1} + \frac{1}{x} \right) \, dx = -2 \ln (x - 1) - \ln x.$$
Hence (9), Sec. 2.1, becomes

\[ u' = U = y_1^{-2} e^{-\int p \, dx} = \frac{(x - 1)^2}{(x - 1)^2 x} = \frac{1}{x}, \quad u = \ln x, \quad y_2 = uy_1 = \frac{\ln x}{1 - x}. \]

\( y_1 \) and \( y_2 \) are shown in Fig. 106. These functions are linearly independent and thus form a basis on the interval \( 0 < x < 1 \) (as well as on \( 1 < x < \infty \)).
Fig. 106. Solutions in Example 2
**EXAMPLE 3** Case 3, Second Solution with Logarithmic Term

Solve the ODE

\[(x^2 - x)y'' - xy' + y = 0.\]

**Solution.** Substituting (2) and (2*) into (13), we have

\[
(x^2 - x) \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-2} - x \sum_{m=0}^{\infty} (m + r)a_m x^{m+r-1} + \sum_{m=0}^{\infty} a_m x^{m+r} = 0.
\]

We now take \(x^2, x,\) and \(x\) inside the summations and collect all terms with power \(x^{m+r}\) and simplify algebraically,

\[
\sum_{m=0}^{\infty} (m + r - 1)^2 a_m x^{m+r} - \sum_{m=0}^{\infty} (m + r)(m + r - 1)a_m x^{m+r-1} = 0.
\]
In the first series we set \( m = s \) and in the second \( m = s + 1 \), thus \( s = m - 1 \). Then

\[
\sum_{s=0}^{\infty} (s + r - 1)^2 a_s x^{s+r} - \sum_{s=1}^{\infty} (s + r + 1)(s + r)a_{s+1} x^{s+r} = 0.
\]

The lowest power is \( x^{r-1} \) (take \( s = -1 \) in the second series) and gives the indicial equation

\[
r(r - 1) = 0.
\]

The roots are \( r_1 = 1 \) and \( r_2 = 0 \). They differ by an integer. This is Case 3.
First Solution. From (14) with \( r = r_1 = 1 \) we have

\[
\sum_{s=0}^{\infty} \left[ s^2 a_s - (s + 2)(s + 1)a_{s+1} \right] x^{s+1} = 0.
\]

This gives the recurrence relation

\[
a_{s+1} = \frac{s^2}{(s + 2)(s + 1)} a_s \quad (s = 0, 1, \cdots).
\]

Hence \( a_1 = 0, \ a_2 = 0, \cdots \) successively. Taking \( a_0 = 1 \), we get as a first solution \( y_1 = x^{r_1}a_0 = x \).
**Second Solution.** Applying reduction of order (Sec. 2.1), we substitute \( y_2 = y_1 u = xu, \ y'_2 = xu' + u \) and \( y''_2 = xu'' + 2u' \) into the ODE, obtaining

\[
(x^2 - x)(xu'' + 2u') - x(xu' + u) + xu = 0.
\]

\( xu \) drops out. Division by \( x \) and simplification give

\[
(x^2 - x)u'' + (x - 2)u' = 0.
\]
From this, using partial fractions and integrating (taking the integration constant zero), we get

\[ \frac{u''}{u'} = -\frac{x-2}{x^2-x} = -\frac{2}{x} + \frac{1}{x-1}, \quad \ln u' = \ln \left| \frac{x-1}{x^2} \right|. \]

Taking exponents and integrating (again taking the integration constant zero), we obtain

\[ u' = \frac{x-1}{x^2} = \frac{1}{x} - \frac{1}{x^2}, \quad u = \ln x + \frac{1}{x}, \quad y_2 = xu = x \ln x + 1. \]

\( y_1 \) and \( y_2 \) are linearly independent, and \( y_2 \) has a logarithmic term. Hence \( y_1 \) and \( y_2 \) constitute a basis of solutions for positive \( x \).
5.5 Bessel’s Equation. 
Bessel Functions $J_{\nu}(x)$

One of the most important ODEs in applied mathematics is Bessel’s equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0.$$ (1)

Its diverse applications range from electric fields to heat conduction and vibrations (see Sec. 12.9). It often appears when a problem shows cylindrical symmetry (just as Legendre’s equation may appear in cases of spherical symmetry). The parameter $\nu$ in (1) is a given number. We assume that $\nu$ is real and nonnegative.
We substitute the series

\[ y(x) = \sum_{m=0}^{\infty} a_m x^{m+r} \quad (a_0 \neq 0) \]

with undetermined coefficients and its derivatives into (1). This gives

\[
\sum_{m=0}^{\infty} (m+r)(m+r-1)a_m x^{m+r} + \sum_{m=0}^{\infty} (m+r)a_m x^{m+r} \\
+ \sum_{m=0}^{\infty} a_m x^{m+r+2} - \nu^2 \sum_{m=0}^{\infty} a_m x^{m+r} = 0.
\]
We obtain the **indicial equation**

\[(4) \quad (r + \nu)(r - \nu) = 0.\]

The roots are \(r_1 = \nu \quad (\geq 0)\) and \(r_2 = -\nu\).

\[(7) \quad a_{2m} = \frac{(-1)^m a_0}{2^m m! (\nu + 1)(\nu + 2) \cdots (\nu + m)}, \quad m = 1, 2, \ldots\]
Bessel Functions $J_n(x)$ For Integer $\nu = n$

Integer values of $\nu$ are denoted by $n$. This is standard. For $\nu = n$ the relation (7) becomes

$$a_{2m} = \frac{(-1)^m a_0}{2^m m! (n + 1)(n + 2) \cdots (n + m)}, \quad m = 1, 2, \cdots$$

$a_0$ is still arbitrary, so that the series (2) with these coefficients would contain this arbitrary factor $a_0$. This would be a highly impractical situation for developing formulas or computing values of this new function. Accordingly, we have to make a choice. $a_0 = 1$ would be possible, but more practical turns out to be
(9) \[ a_0 = \frac{1}{2^n n!} \cdot \]

because then \( n!(n + 1) \cdots (n + m) = (m + n)! \) in (8), so that (8) simply becomes

(10) \[ a_{2m} = \frac{(-1)^m}{2^{2m+n} m! (n + m)!}, \quad m = 1, 2, \cdots. \]

This simplicity of the denominator of (10) partially motivates the choice (9). With these coefficients and \( r_1 = \nu = n \) we get from (2) a particular solution of (1), denoted by \( J_n(x) \) and given by
Chapter 5 Series Solutions of ODEs. Special Functions

\[ J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m! (n + m)!}. \]

\( J_n(x) \) is called the **Bessel function of the first kind of order** \( n \). The series (11) converges for all \( x \), as the ratio test shows. In fact, it converges very rapidly because of the factorials in the denominator.
EXAMPLE 1  Bessel Functions $J_0(x)$ and $J_1(x)$

For $n = 0$ we obtain from (11) the Bessel function of order 0

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} = 1 - \frac{x^2}{2^2(1!)^2} + \frac{x^4}{2^4(2!)^2} - \frac{x^6}{2^6(3!)^2} + \cdots$$

which looks similar to a cosine (Fig. 107). For $n = 1$ we obtain the Bessel function of order 1

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{2^{2m+1}m! (m + 1)!} = \frac{x}{2} - \frac{x^3}{2^31!2!} + \frac{x^5}{2^52!3!} - \frac{x^7}{2^73!4!} + \cdots.$$
which looks similar to a sine (Fig. 107). But the zeros of these functions are not completely regularly spaced (see also Table A1 in App. 5) and the height of the “waves” decreases with increasing $x$. Heuristically, $n^2/x^2$ in (1) in standard form [(1) divided by $x^2$] is zero (if $n = 0$) or small in absolute value for large $x$, and so is $y'/x$, so that then Bessel’s equation comes close to $y'' + y = 0$, the equation of $\cos x$ and $\sin x$; also $y'/x$ acts as a “damping term,” in part responsible for the decrease in height. One can show that for large $x$,

$$J_n(x) \sim \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{n\pi}{2} - \frac{\pi}{4} \right)$$
where ~ is read “asymptotically equal” and means that for fixed $n$ the quotient of the two sides approaches 1 as $x \to \infty$.

Formula (14) is surprisingly accurate even for smaller $x (> 0)$. For instance, it will give you good starting values in a computer program for the basic task of computing zeros. For example, for the first three zeros of $J_0$ you obtain the values 2.356 (2.405 exact to 3 decimals, error 0.049), 5.498 (5.520, error 0.022), 8.639 (8.654, error 0.015), etc.
Fig. 107. Bessel functions of the first kind $J_0$ and $J_1$
Bessel Functions $J_\nu(x)$ for any $\nu \geq 0$.  

**Gamma Function**

Hence because of our (standard!) choice (18) of $a_0$ the coefficients (7) simply are

\[
a_{2m} = \frac{(-1)^m}{2^{2m+\nu} m! \Gamma(\nu + m + 1)}.\]

(19)

With these coefficients and $r = r_1 = \nu$ we get from (2) a particular solution of (1), denoted by $J_\nu(x)$ and given by

\[
J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+\nu} m! \Gamma(\nu + m + 1)}.\]

(20)

$J_\nu(x)$ is called the **Bessel function of the first kind of order** $\nu$. The series (20) converges for all $x$, as one can verify by the ratio test.
General Solution for Noninteger $\nu$. Solution $J_{-\nu}$

**THEOREM 1**

General Solution of Bessel’s Equation

If $\nu$ is not an integer, a general solution of Bessel’s equation for all $x \neq 0$ is

\[
y(x) = c_1 J_{\nu}(x) + c_2 J_{-\nu}(x).\]
THEOREM 2

Linear Dependence of Bessel Functions $J_n$ and $J_{-n}$

For integer $\nu = n$ the Bessel functions $J_n(x)$ and $J_{-n}(x)$ are linearly dependent, because

$$J_{-n}(x) = (-1)^n J_n(x) \quad (n = 1, 2, \cdots).$$
THEOREM 3

**Derivatives, Recursions**

The derivative of $J_{\nu}(x)$ with respect to $x$ can be expressed by $J_{\nu-1}(x)$ or $J_{\nu+1}(x)$ by the formulas

(a) $[x^{\nu}J_{\nu}(x)]' = x^{\nu}J_{\nu-1}(x)$

(b) $[x^{-\nu}J_{\nu}(x)]' = -x^{-\nu}J_{\nu+1}(x)$.

Furthermore, $J_{\nu}(x)$ and its derivative satisfy the recurrence relations

(c) $J_{\nu-1}(x) + J_{\nu+1}(x) = \frac{2\nu}{x}J_{\nu}(x)$

(d) $J_{\nu-1}(x) - J_{\nu+1}(x) = 2J_{\nu}'(x)$. 
EXAMPLE 2  Application of Theorem 3 in Evaluation and Integration

Formula (24c) can be used recursively in the form

\[ J_{\nu+1}(x) = \frac{2\nu}{x} J_\nu(x) - J_{\nu-1}(x) \]

for calculating Bessel functions of higher order from those of lower order. For instance, \( J_2(x) = \frac{2J_1(x)}{x} - J_0(x) \), so that \( J_2 \) can be obtained from tables of \( J_0 \) and \( J_1 \) (in App. 5 or, more accurately, in Ref. [GR1] in App. 1).
To illustrate how Theorem 3 helps in integration, we use (24b) with $\nu = 3$ integrated on both sides. This evaluates, for instance, the integral

$$I = \int_1^2 x^{-3} J_4(x) \, dx = -x^{-3} J_3(x) \bigg|_1^2 = -\frac{1}{8} J_3(2) + J_3(1).$$

A table of $J_3$ (of Ref. [GR1]) or your CAS will give you

$$-\frac{1}{8} \cdot 0.128943 + 0.019563 = 0.003445.$$
Your CAS (or a human computer in precomputer times) obtains $J_3$ from (24), first using (24c) with $\nu = 2$, that is, $J_3 = 4x^{-1}J_2 - J_1$ then (24c) with $\nu = 1$, that is, $J_2 = 2x^{-1}J_1 - J_0$. Together,

$$I = x^{-3}(4x^{-1}(2x^{-1}J_1 - J_0) - J_1) \bigg|^{2}_1$$

$$= -\frac{1}{8}[2J_1(2) - 2J_0(2) - J_1(2)] + [8J_1(1) - 4J_0(1) - J_1(1)]$$

$$= -\frac{1}{8}J_1(2) + \frac{1}{4}J_0(2) + 7J_1(1) - 4J_0(1).$$

This is what you get, for instance, with Maple if you type \texttt{int(...)}. And if you type \texttt{evalf(int(...))}, you obtain 0.003445448, in agreement with the result near the beginning of the example.
THEOREM 4

Elementary $J_\nu$ for Half-Integer Order $\nu$

Bessel functions $J_\nu$ of orders $\pm 1/2$, $\pm 3/2$, $\pm 5/2$ \ldots are elementary; they can be expressed by finitely many cosines and sines and powers of $x$. In particular,

(25) \hspace{1cm}

\begin{align*}
(a) \quad J_{1/2}(x) &= \sqrt{\frac{2}{\pi x}} \sin x, \\
(b) \quad J_{-1/2}(x) &= \sqrt{\frac{2}{\pi x}} \cos x.
\end{align*}
EXAMPLE 3  Further Elementary Bessel Functions

From (24c) with \( \nu = 1/2 \) and \( \nu = -1/2 \) and (25) we obtain

\[
J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \left( \frac{\sin x}{x} - \cos x \right)
\]

\[
J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x) = -\sqrt{\frac{2}{\pi x}} \left( \frac{\cos x}{x} + \sin x \right)
\]

respectively, and so on.
5.6 Bessel Functions of the Second Kind $Y_\nu(x)$

$n = 0$: Bessel Function of the Second Kind $Y_0(x)$

When $n = 0$, Bessel’s equation can be written

$$xy'' + y' + xy = 0. \tag{1}$$

Then the indicial equation (4) in Sec. 5.5 has a double root $r = 0$. This is Case 2 in Sec. 5.4. In this case we first have only one solution, $J_0(x)$. From (8) in Sec. 5.4 we see that the desired second solution must be of the form

$$y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} A_m x^m. \tag{2}$$
Using the short notations

\[ h_1 = 1 \quad h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m} \quad m = 2, 3, \cdots \]

and inserting (4) and \( A_1 = A_3 = \cdots = 0 \) into (2), we obtain the result

\[ y_2(x) = J_0(x) \ln x + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^m (m!)^2} x^{2m} \]

(5)

\[ = J_0(x) \ln x + \frac{1}{4} x^2 - \frac{3}{128} x^4 + \frac{11}{13824} x^6 + \cdots. \]
Since $J_0$ and $y_2$ are linearly independent functions, they form a basis of (1) for $x > 0$. Of course, another basis is obtained if we replace $y_2$ by an independent particular solution of the form $a(y_2 + bJ_0)$, where $a \neq 0$ and $b$ are constants. It is customary to choose $a = 2\pi$ and $b = \gamma - \ln 2$, where the number $\gamma = 0.57721566490\cdots$ is the so-called Euler constant, which is defined as the limit of

$$1 + \frac{1}{2} + \cdots + \frac{1}{s} - \ln s$$
as $s$ approaches infinity. The standard particular solution thus obtained is called the **Bessel function of the second kind of order zero** (Fig. 109) or **Neumann’s function of order zero** and is denoted by $Y_0(x)$. Thus [see (4)]

\[
Y_0(x) = \frac{2}{\pi} \left[ J_0(x) \left( \ln \frac{x}{2} + \gamma \right) + \sum_{m=1}^{\infty} \frac{(-1)^{m-1} h_m}{2^{2m}(m!)^2} x^{2m} \right].
\]

For small $x > 0$ the function $Y_0(x)$ behaves about like $\ln x$ (see Fig. 109, why?), and $Y_0(x) \rightarrow -\infty$ as $x \rightarrow 0$. 
Bessel Functions of the Second Kind $Y_n(x)$

For this reason we introduce a standard second solution $Y_n(x)$ defined for all $\nu$ by the formula

\[ Y_\nu(x) = \frac{1}{\sin \nu\pi} \left( J_\nu(x) \cos \nu\pi - J_{-\nu}(x) \right) \]

(7)

This function is called the Bessel function of the second kind of order $\nu$ or Neumann’s function of order $\nu$. Figure 109 shows $Y_0(x)$ and $Y_1(x)$.
Fig. 109. Bessel functions of the second kind $Y_0$ and $Y_1$. (For a small table, see App. 5.)
where \( x > 0, \ n = 0, 1, \cdots, \) and [as in (4)] \( h_0 = 0, \ h_1 = 1, \)

\[
\begin{align*}
\frac{2}{\pi} J_n(x) \left( \ln \frac{x}{2} + \gamma \right) + \frac{x^n}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m-1}(h_m + h_{m+n})}{2^{2m+n}m!(m+n)!} x^{2m} \\
- \frac{x^{-n}}{\pi} \sum_{m=0}^{n-1} \frac{(n-m-1)!}{2^{2m-n}m!} x^{2m}
\end{align*}
\]

\[
h_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}, \quad h_{m+n} = 1 + \frac{1}{2} + \cdots + \frac{1}{m+n}.
\]
THEOREM 1

General Solution of Bessel’s Equation

A general solution of Bessel’s equation for all values of $\nu$ (and $x > 0$) is

(9) $y(x) = C_1 J_\nu(x) + C_2 Y_\nu(x)$. 
So far we have considered initial value problems. We recall from Sec. 2.1 that such a problem consists of an ODE, say, of second order, and initial conditions \( y(x_0) = K_0, \ y'(x_0) = K_1 \) referring to the same point (initial point) \( x = x_0 \). We now turn to boundary value problems. A boundary value problem consists of an ODE and given boundary conditions referring to the two boundary points (endpoints) \( x = a \) and \( x = b \) of a given interval \( a \leq x \leq b \). To solve such a problem means to find a solution of the ODE on the interval \( a \leq x \leq b \) satisfying the boundary conditions.
We shall see that Legendre’s, Bessel’s, and other ODEs of importance in engineering can be written as a Sturm–Liouville equation

\[ [p(x)y']' + [q(x) + \lambda r(x)]y = 0 \]

involving a parameter \( \lambda \). The boundary value problem consisting of an ODE (1) and given Sturm–Liouville boundary conditions

(a) \( k_1 y(a) + k_2 y'(a) = 0 \)

(b) \( l_1 y(b) + l_2 y'(b) = 0 \)
is called a **Sturm–Liouville problem**. We shall see further that these problems lead to useful series developments in terms of particular solutions of (1), (2). Crucial in this connection is **orthogonality** to be discussed later in this section.

In (1) we make the **assumptions** that \( p, q, r, \) and \( p' \) are continuous on \( a \leq x \leq b \), and

\[
  r(x) > 0 \quad (a \leq x \leq b).
\]

In (2) we assume that \( k_1, k_2 \) are given constants, not both zero, and so are \( l_1, l_2 \), not both zero.
EXAMPLE 1  Legendre’s and Bessel’s Equations are Sturm–Liouville Equations

Legendre’s equation \((1 – x^2)y'' – 2xy' + n(n + 1)y = 0\) may be written

\[
[(1 – x^2)y']’ + \lambda y = 0 \quad \quad \lambda = n(n + 1).
\]

This is (1) with \(p = 1 – x^2, \ q = 0, \) and \(r = 1.\)

In Bessel’s equation

\[
\ddot{\tilde{x}}^2 \ddot{y} + \dot{\tilde{x}} \dot{y} + (\tilde{x}^2 - n^2)y = 0 \quad \quad \dot{\tilde{y}} = dy/d\tilde{x}, \quad \text{etc.}
\]
as a model in physics or elsewhere, one often likes to have another parameter $k$ in addition to $n$. For this reason we set $\tilde{x} = kx$. Then by the chain rule $\dot{y} = dy/d\tilde{x} = (dy/dx) \cdot dy/d\tilde{x} = y'/k$, $\ddot{y} = y''/k^2$. In the first two terms, $k^2$ and $k$ drop out and we get

$$x^2 y'' + xy' + (k^2 x^2 - n^2) y = 0.$$ 

Division by $x$ gives the Sturm–Liouville equation

$$[xy']' + \left( -\frac{n^2}{x} + \lambda x \right) y = 0$$

This is (1) with $p = x$, $q = -n^2/x$, and $r = x$. 

\[ \lambda = k^2. \]
Clearly, \( y \equiv 0 \) is a solution—the "trivial solution"—for any \( \lambda \) because (1) is homogeneous and (2) has zeros on the right. This is of no interest. We want to find eigenfunctions \( y(x) \), that is, solutions of (1) satisfying (2) without being identically zero. We call a number \( \lambda \) for which an eigenfunction exists an eigenvalue of the Sturm–Liouville problem (1), (2).
EXAMPLE 2 Trigonometric Functions as Eigenfunctions. Vibrating String

Find the eigenvalues and eigenfunctions of the Sturm–Liouville problem

\[ y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0. \]

This problem arises, for instance, if an elastic string (a violin string, for example) is stretched a little and then fixed at its ends \( x = 0 \) and \( x = \pi \) and allowed to vibrate. Then \( y(x) \) is the “space function” of the deflection \( u(x, t) \) of the string, assumed in the form \( u(x, t) = y(x)w(t) \), where \( t \) is time. (This model will be discussed in great detail in Secs. 12.2–12.4.)
**Solution.** From (1) and (2) we see that \( p = 1, \ q = 0, \ r = 1 \) in (1), and \( a = 0, \ b = \pi, \ k_1 = l_1 = 1, \ k_2 = l_2 = 0 \) in (2). For negative \( \lambda = -\nu^2 \) a general solution of the ODE in (3) is \( y(x) = c_1 e^{\nu x} + c_2 e^{-\nu x} \). From the boundary conditions we obtain \( c_1 = c_2 = 0 \), so that \( y \equiv 0 \), which is not an eigenfunction. For \( \lambda = 0 \) the situation is similar. For positive \( \lambda = \nu^2 \) a general solution is

\[
y(x) = A \cos \nu x + B \sin \nu x.
\]

From the first boundary condition we obtain \( y(0) = A = 0 \). The second boundary condition then yields

\[
y(\pi) = B \sin \nu \pi = 0, \quad \text{thus} \quad \nu = 0, \pm1, \pm2, \ldots \ldots.
\]
For $\nu = 0$ we have $y \equiv 0$. For $\lambda = \nu^2 = 1, 4, 9, 16, \cdots$, taking $B = 1$, we obtain

$$y(x) = \sin \nu x \quad (\nu = 1, 2, \cdots).$$

Hence the eigenvalues of the problem are $\lambda = \nu^2$, where $\nu = 1, 2, \cdots$, and corresponding eigenfunctions are $y(x) = \sin \nu x$, where $\nu = 1, 2, \cdots$. 

Chapter 5 Series Solutions of ODEs. Special Functions
Orthogonality

DEFINITION

Orthogonality (1)

Functions \( y_1(x), y_2(x), \ldots \) defined on some interval \( a \leq x \leq b \) are called orthogonal on this interval with respect to the weight function \( r(x) > 0 \) if for all \( m \) and all \( n \) different from \( m \),

\[
\int_{a}^{b} r(x) y_m(x) y_n(x) \, dx = 0 \quad (m \neq n).
\]

The norm \( \| y_m \| \) of \( y_m \) is defined by

\[
\| y_m \| = \sqrt{\int_{a}^{b} r(x) y_m^2(x) \, dx}.
\]
Orthogonality(2)

Note that this is the square root of the integral in (4) with $n = m$.

The functions $y_1, y_2, \cdots$ are called orthonormal on $a \leq x \leq b$ if they are orthogonal on this interval and all have norm 1.

If $r(x) = 1$, we more briefly call the functions orthogonal instead of orthogonal with respect to $r(x) = 1$; similarly for orthonormality. Then

$$\int_a^b y_m(x) y_n(x) \ dx = 0 \ (m \neq n), \quad \|y_m\| = \sqrt{\int_a^b y_m^2(x) \ dx}.$$
EXAMPLE 3 Orthogonal Functions. Orthonormal Functions

The functions $y_m(x) = \sin mx, \ m = 1, 2,$ form an orthogonal set on the interval $-\pi \leq x \leq \pi$, because for $m \neq n$ we obtain by integration [see (11) in App. A3.1]

$$
\int_{-\pi}^{\pi} y_n(x) y_m(x) \, dx = \int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \cos (m-n)x \, dx - \frac{1}{2} \int_{-\pi}^{\pi} \cos (m+n)x \, dx = 0.
$$

The norm $\|y_m\|$ equals $\sqrt{\pi}$, because

$$
\|y_m\|^2 = \int_{-\pi}^{\pi} \sin^2 mx \, dx = \pi \quad (m = 1, 2, \cdots).
$$

Hence the corresponding orthonormal set, obtained by division by the norm, is

$$
\frac{\sin x}{\sqrt{\pi}}, \quad \frac{\sin 2x}{\sqrt{\pi}}, \quad \frac{\sin 3x}{\sqrt{\pi}}, \quad \cdots.
$$
Orthogonality of Eigenfunctions

**THEOREM 1**

Suppose that the functions \( p, q, r, \) and \( p' \) in the Sturm–Liouville equation (1) are real-valued and continuous and \( r(x) > 0 \) on the interval \( a \leq x \leq b \). Let \( y_m(x) \) and \( y_n(x) \) be eigenfunctions of the Sturm–Liouville problem (1), (2) that correspond to different eigenvalues \( \lambda_m \) and \( \lambda_n \), respectively. Then \( y_m, y_n \) are orthogonal on that interval with respect to the weight function \( r \), that is,

\[
\int_{a}^{b} r(x)y_m(x)y_n(x) \, dx = 0 \quad \text{for} \quad m \neq n.
\]
THEOREM 1

Orthogonality of Eigenfunctions

If \( p(a) = 0 \), then (2a) can be dropped from the problem. If \( p(b) = 0 \), then (2b) can be dropped. [It is then required that \( y \) and \( y' \) remain bounded at such a point, and the problem is called singular, as opposed to a regular problem in which (2) is used.]

If \( p(a) = p(b) \), then (2) can be replaced by the “periodic boundary conditions”

\[
y(a) = y(b), \quad y'(a) = y'(b). \tag{7}
\]
**EXAMPLE 4** Application of Theorem 1.  
Vibrating Elastic String

The ODE in Example 2 is a Sturm–Liouville equation with $p = 1$, $q = 0$, and $r = 1$. From Theorem 1 it follows that the eigenfunctions $y_m = \sin mx$ ($m = 1, 2, \cdots$) are orthogonal on the interval $0 \leq x \leq \pi$. 
E X A M P L E 5  Application of Theorem 1.

Orthogonality of the Legendre Polynomials

Legendre’s equation is a Sturm–Liouville equation (see Example 1)

\[(1 - x^2)y' + \lambda y = 0, \quad \lambda = n(n + 1)\]

with \(p = 1 - x^2, q = 0,\) and \(r = 1.\) Since \(p(-1) = p(1) = 0,\) we need no boundary conditions, but have a singular Sturm—Liouville problem on the interval \(-1 \leq x \leq 1.\) We know that for \(n = 0, 1, \cdots\), hence \(\lambda = 0, 1 \cdot 2, 2 \cdot 3, \cdots\), the Legendre polynomials \(P_n(x)\) are solutions of the problem. Hence these are the eigenfunctions. From Theorem 1 it follows that they are orthogonal on that interval, that is.

\[
\int_{-1}^{1} P_m(x) P_n(x) \, dx = 0 \quad (m \neq n).
\]
Chapter 5 Series Solutions of ODEs. Special Functions

**Example 6** Application of Theorem 1. Orthogonality of the Bessel Functions \( J_n(x) \)

The Bessel function \( J_n(\tilde{x}) \) with fixed integer \( n \geq 0 \) satisfies Bessel’s equation (Sec. 5.5)

\[
\tilde{x}^2 \dddot{J}_n(\tilde{x}) + \tilde{x} \ddot{J}_n(\tilde{x}) + (\tilde{x}^2 - n^2) J_n(\tilde{x}) = 0,
\]

where \( \dot{J}_n = \frac{dJ_n}{d\tilde{x}}, \quad \ddot{J}_n = \frac{d^2J_n}{d\tilde{x}^2} \). In Example 1 we transformed this equation, by setting \( \tilde{x} = kx \), into a Sturm–Liouville equation

\[
[xJ_n'(kx)]' + \left( -\frac{n^2}{x} + k^2x \right) J_n(kx) = 0
\]
with \( p(x) = x \), \( q(x) = -n^2/x \), \( r(x) = x \), and parameter \( \lambda = k^2 \). Since \( p(0) = 0 \), Theorem 1 implies orthogonality on an interval \( 0 \leq x \leq R \) (\( R \) given, fixed) of those solutions \( J_n(kx) \) that are zero at \( x = R \), that is,

\[
J_n(kR) = 0 \quad \text{(} n \text{ fixed).}
\]

[Note that \( q(x) = -n^2/x \) is discontinuous at 0, but this does not affect the proof of Theorem 1.] It can be shown (see Ref. [A13]) that \( J_n(\tilde{x}) \) has infinitely many zeros, say, \( \tilde{x} = \alpha_{n,1} < \alpha_{n,2} \) (see Fig. 107 in Sec. 5.5 for \( n = 0 \) and 1). Hence we must have

\[
kR = \alpha_{n,m} \quad \text{thus} \quad k_{n,m} = \frac{\alpha_{n,m}}{R} \quad (m = 1, 2, ).
\]

This proves the following orthogonality property.
THEOREM 2

Orthogonality of Bessel Functions

For each fixed nonnegative integer \( n \) the sequence of Bessel functions of the first kind \( J_n(k_{n,1}x), \ J_n(k_{n,2}x), \ \cdots \) with \( k_{n,m} \) as in (12) forms an orthogonal set on the interval \( 0 \leq x \leq R \) with respect to the weight function \( r(x) = x \), that is,

\[
\int_0^R xJ_n(k_{n,m}x)J_n(k_{n,j}x) \, dx = 0 \quad (j \neq m, \ n \text{ fixed}).
\]
**Example 7** Eigenvalues from Graphs

Solve the Sturm–Liouville problem $y'' + \lambda y = 0$, $y(0) + y'(0) = 0$, $y(\pi) - y'(\pi) = 0$.

**Solution.** A general solution and its derivative are

$$y = A \cos kx + B \sin kx \quad \text{and} \quad y' = -Ak \sin kx + Bk \cos kx, \quad k = \sqrt{\lambda}.$$  

The first boundary condition gives $y(0) + y'(0) = A + Bk = 0$, hence $A = -Bk$. The second boundary condition and substitution of $A = -Bk$ give

$$y(\pi) - y'(\pi)$$
$$= A \cos \pi k + B \sin \pi k + Ak \sin \pi k - Bk \cos \pi k$$
$$= -Bk \cos \pi k + B \sin \pi k - Bk^2 \sin \pi k - Bk \cos \pi k$$
$$= 0$$
We must have $B \neq 0$ since otherwise $B = A = 0$, hence $y = 0$, which is not an eigenfunction. Division by $B \cos \pi k$ gives

$$-k + \tan \pi k - k^2 \tan \pi k - k = 0,$$

thus

$$\tan \pi k = \frac{-2k}{k^2 - 1}.$$ 

The graph in Fig. 110 now shows us where to look for eigenvalues. These correspond to the $k$-values of the points of intersection of $\tan \pi k$ and the right side $-2k/(k^2 - 1)$ of the last equation. The eigenvalues are $\lambda_m = k_m^2$, where $\lambda_0 = 0$ with eigenfunction $y_0 = 1$ and the other $\lambda_m$ are located near $2^2, 3^2, 4^2, \ldots$, with eigenfunctions $\cos k_m x$ and $\sin k_m x$, $m = 1, 2, \ldots$. The precise numeric determination of the eigenvalues would require a root-finding method (such as those given in Sec. 19.2).
Fig. 110. Example 7. Circles mark the intersections of $\tan \pi k$ and $-2k/(k^2 - 1)$.
Kronecker’s delta $\delta_{mn}$ is defined by $\delta_{mn} = 0$ if $m \neq n$ and $\delta_{mn} = 1$ if $m = n$ (thus $\delta_{nn} = 1$). Hence for orthonormal functions $y_0, y_1, y_2, \ldots$ with respect to weight $r(x) (> 0)$ on $a \leq x \leq b$ we can now simply write $(y_m, y_n) = \delta_{mn}$, written out

$$\begin{align*}
(y_m, y_n) &= \int_a^b r(x)y_m(x)y_n(x) \, dx = \delta_{mn} = \begin{cases} 
0 & \text{if } m \neq n \\
1 & \text{if } m = n.
\end{cases}
\end{align*}$$

Also, for the norm we can now write

$$\begin{align*}
\|y\| &= \sqrt{(y_m, y_m)} = \sqrt{\int_a^b r(x)y_m^2(x) \, dx}.
\end{align*}$$
Orthogonal Series

Let \( y_0, y_1, y_2, \ldots \) be an orthogonal set with respect to weight \( r(x) \) on an interval \( a \leq x \leq b \). Let \( f(x) \) be a function that can be represented by a convergent series

\[
(3) \\
 f(x) = \sum_{m=0}^{\infty} a_m y_m(x) = a_0 y_0(x) + a_1 y_1(x) + \cdots.
\]

This is called an orthogonal expansion or generalized Fourier series. If the \( y_m \) are eigenfunctions of a Sturm–Liouville problem, we call (3) an eigenfunction expansion.

Then we obtain
Because of the orthogonality all the integrals on the right are zero, except when \( m = n \). Hence the whole infinite series reduces to the single term

\[
a_n(y_n, y_n) = a_n \| y_n \|^2.
\]

Assuming that all the functions \( y_n \) have nonzero norm, we can divide by \( \| y_n \|^2 \); writing again \( m \) for \( n \), to be in agreement with (3), we get the desired formula for the Fourier constants

\[
a_m = \frac{(f, y_m)}{\| y_m \|^2} = \frac{1}{\| y_m \|^2} \int_a^b r(x)f(x)y_m(x) \, dx \quad (m = 0, 1, \cdots).
\]
EXAMPLE 1 Fourier Series

A most important class of eigenfunction expansions is obtained from the periodic Sturm–Liouville problem

\[ y'' + \lambda y = 0, \quad y(\pi) = y(-\pi), \quad y'(\pi) = y'(-\pi). \]

A general solution of the ODE is \( y = A \cos kx + B \sin kx \), where \( k = \). Substituting \( y \) and its derivative into the boundary conditions, we obtain

\[ A \cos k\pi + B \sin k\pi = A \cos (-k\pi) + B \sin (-k\pi) \]

\[ -kA \sin k\pi + kB \cos k\pi = -kA \sin (-k\pi) + kB \cos (-k\pi). \]
Since \( \cos (-\alpha) = \cos \alpha \), the cosine terms cancel, so that these equations give no condition for these terms. Since \( \sin (-\alpha) = -\sin \alpha \), the equations gives the condition \( \sin k\pi = 0 \), hence \( k\pi = m\pi \), \( k = m = 0, 1, 2, \cdots \), so that the eigenfunctions are

\[
\cos 0 = 1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots , \cos mx, \sin mx, \cdots
\]

corresponding pairwise to the eigenvalues \( \lambda = k^2 = 0, 1, 4, \cdots, m^2, \cdots \). (\( \sin 0 = 0 \) is not an eigenfunction.)
By Theorem 1 in Sec. 5.7, any two of these belonging to different eigenvalues are orthogonal on the interval \(-\pi \leq x \leq \pi\) (note that \(r(x) = 1\) for the present ODE). The orthogonality of \(\cos mx\) and \(\sin mx\) for the same \(m\) follows by integration,

\[
\int_{-\pi}^{\pi} \cos mx \sin mx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} \sin 2mx \, dx = 0.
\]

For the norms we get \(\|1\| = \sqrt{2\pi}\), and \(\sqrt{\pi}\) for all the others, as you may verify by integrating \(1, \cos^2 x, \sin^2 x, \) etc., from \(-\pi\) to \(\pi\). This gives the series (with a slight extension of notation since we have two functions for each eigenvalue \(1, 4, 9, \ldots\))

\[
f(x) = a_0 + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx).
\]
According to (4) the coefficients (with \( m = 1, 2, \cdots \) ) are

\[
a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx, \quad a_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx \, dx, \quad b_m = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx \, dx.
\]

The series (5) is called the **Fourier series** of \( f(x) \). Its coefficients are called the **Fourier coefficients** of \( f(x) \), as given by the so-called **Euler formulas** (6) (not to be confused with the Euler formula (11) in Sec. 2.2).

For instance, for the “**periodic rectangular wave**” in Fig. 111, given by

\[
f(x) = \begin{cases} 
-1 & \text{if } -\pi < x < 0 \\
1 & \text{if } 0 < x < \pi 
\end{cases}
\quad \text{and} \quad f(x + 2\pi) = f(x).
\]
we get from (6) the values $a_0 = 0$ and

\[ a_m = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-1) \cos mx \, dx + \int_{0}^{\pi} 1 \cdot \cos mx \, dx \right] = 0, \]

\[ b_m = \frac{1}{\pi} \left[ \int_{-\pi}^{0} (-1) \sin mx \, dx + \int_{0}^{\pi} 1 \cdot \sin mx \, dx \right] \]

\[ = \frac{1}{\pi} \left[ \frac{\cos mx}{m} \right]_{-\pi}^{0} - \frac{\cos mx}{m} \bigg|_{0}^{\pi} \]

\[ = \frac{1}{\pi m} [1 - 2 \cos m\pi + 1] = \begin{cases} 4/(\pi m) & \text{if } m = 1, 3, \ldots, \\ 0 & \text{if } m = 2, 4 \ldots. \end{cases} \]

Hence the Fourier series of the periodic rectangular wave is

\[ f(x) = \frac{4}{\pi} \left( \sin x + \frac{1}{3} \sin 3x + \frac{1}{5} \sin 5x + \cdots \right). \]
Fig. 111. Periodic rectangular wave in Example 1
EXAMPLE 2  Fourier–Legendre Series

A Fourier–Legendre series is an eigenfunction expansion

\[ f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 P_0 + a_1 P_1(x) + a_2 P_2(x) + \cdots = a_0 + a_1 x + a_2 \left( \frac{3}{2} x^2 - \frac{1}{2} \right) + \cdots \]

in terms of Legendre polynomials (Sec. 5.3). The latter are the eigenfunctions of the Sturm–Liouville problem in Example 5 of Sec. 5.7 on the interval \(-1 \leq x \leq 1\). We have \( r(x) = 1 \) for Legendre’s equation, and (4) gives

\[ a_m = \frac{2m + 1}{2} \int_{-1}^{1} f(x) P_m(x) \, dx, \quad m = 0, 1, \cdots \]
because the norm is

\[ \| P_m \| = \sqrt{\int_{-1}^{1} P_m(x)^2 \, dx} = \sqrt{\frac{2}{2m + 1}} \quad (m = 0, 1, \cdots) \]  

as we state without proof. (The proof is tricky; it uses Rodrigues’s formula in Problem Set 5.3 and a reduction of the resulting integral to a quotient of gamma functions.)

For instance, let \( f(x) = \sin \pi x \). Then we obtain the coefficients

\[ a_m = \frac{2m + 1}{2} \int_{-1}^{1} (\sin \pi x) P_m(x) \, dx, \quad \text{thus} \quad a_1 = \frac{3}{2} \int_{-1}^{1} x \sin \pi x \, dx = \frac{3}{\pi} = 0.95493, \]
Hence the Fourier–Legendre series of \( \sin \pi x \) is

\[
\sin \pi x = 0.95493 P_1(x) - 1.15824 P_3(x) + 0.21429 P_5(x) - 0.01664 P_7(x) + 0.00068 P_9(x) - 0.00002 P_{11}(x) + \cdots.
\]

The coefficient of \( P_{13} \) is about \( 3 \times 10^{-7} \). The sum of the first three nonzero terms gives a curve that practically coincides with the sine curve. Can you see why the even-numbered coefficients are zero? Why \( a_3 \) is the absolutely biggest coefficient?
EXAMPLE 3 Fourier–Bessel Series

In Example 6 of Sec. 5.7 we obtained infinitely many orthogonal sets of Bessel functions, one for each of \( J_0, J_1, J_2, \ldots \). Each set is orthogonal on an interval \( 0 \leq x \leq R \) with a fixed positive \( R \) of our choice and with respect to the weight \( x \). The orthogonal set for \( J_n \) is \( J_n(k_{n,1}x), J_n(k_{n,2}x), J_n(k_{n,3}x), \ldots \), where \( n \) is fixed and \( k_{n,m} \) is given in (12), Sec. 5.7. The corresponding Fourier–Bessel series is

\[
(9) \quad f(x) = \sum_{m=1}^{\infty} a_m J_n(k_{n,m}x) = a_1 J_n(k_{n,1}x) + a_2 J_n(k_{n,2}x) + a_3 J_n(k_{n,3}x) + \cdots \quad (n \text{ fixed}).
\]
The coefficients are (with $\alpha_{n,m} = k_{n,m}R$)

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x f(x) J_n(k_{n,m}x) \, dx,$$

where $m = 1, 2, \cdots$ (10)

because the square of the norm is

$$\|J_n(k_{n,m}x)\|^2 = \int_0^R x J_n^2(k_{n,m}x) \, dx = \frac{R^2}{2} J_{n+1}^2(k_{n,m}R)$$

as we state without proof (which is tricky; see the discussion beginning of [A13]).
For instance, let us consider \( f(x) = 1 - x^2 \) and take \( R = 1 \) and \( n = 0 \) in the series (9), simply writing \( \lambda \) for \( \alpha_{0,m} \). Then \( k_{n,m} = \alpha_{0,m} = \lambda = 2.405, 5.520, 8.654, 11.792, \) etc. (use a CAS or Table A1 in App. 5). Next we calculate the coefficients \( a_m \) by (10),

\[
a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1 - x^2)J_0(\lambda x) \, dx.
\]

This can be integrated by a CAS or by formulas as follows. First use \([xJ_1(\lambda x)]' = \lambda xJ_0(\lambda x)\) from Theorem 3 in Sec. 5.5 and then integration by parts.

\[
a_m = \frac{2}{J_1^2(\lambda)} \int_0^1 x(1 - x^2)J_0(\lambda x) \, dx = \frac{2}{J_1^2(\lambda)} \left[ \frac{1}{\lambda} (1 - x^2)xJ_1(\lambda x) \right]_0^1 - \frac{1}{\lambda} \int_0^1 xJ_1(\lambda x)(-2x) \, dx.
\]
The integral-free part is zero. The remaining integral can be evaluated by \[ x^2 J_2(\lambda x) = \lambda x^2 J_1(\lambda x) \] from Theorem 3 in Sec. 5.5. This gives

\[
a_m = \frac{4J_2(\lambda)}{\lambda^2 J_1^2(\lambda)} \quad (\lambda = \alpha_{0,m}).
\]

Numeric values can be obtained from a CAS (or from the table of Ref. [GR1] in App. 1, together with the formula \( J_2 = 2x^{-1}J_1 - J_0 \) in Theorem 3 of Sec. 5.5). This gives the eigenfunction expansion of \( 1 - x^2 \) in terms of Bessel functions \( J_0 \), that is,

\[
1 - x^2 = 1.1081J_0(2.405x) - 0.1398J_0(5.520x) + 0.0455J_0(8.654x) - 0.0210J_0(11.792x) + \cdots.
\]

A graph would show that the curve of \( 1 - x^2 \) and that of the sum of the first three terms practically coincide.
Mean Square Convergence.  
Completeness of Orthonormal Sets

Convergence is **convergence in the norm**, also called **mean-square convergence**; that is, a sequence of functions $f_k$ is called **convergent with the limit** $f$ if

\[
\lim_{k \to \infty} \| f_k - f \| = 0; 
\]

written out by (2) (where we can drop the square root, as this does not affect the limit)

\[
(12) \quad \lim_{k \to \infty} \int_a^b r(x)[f_k(x) - f(x)]^2 \, dx = 0. 
\]

Accordingly, the series (3) converges and represents $f$ if

\[
(13) \quad \lim_{k \to \infty} \int_a^b r(x)[s_k(x) - f(x)]^2 \, dx = 0
\]
where $s_k$ is the $k$th partial sum of (3),

\begin{equation}
    s_k(x) = \sum_{m=0}^{k} a_m y_m(x).
\end{equation}

By definition, an orthonormal set $y_0, y_1, \ldots$ on an interval $a \leq x \leq b$ is **complete in a set of functions** $S$ defined on $a \leq x \leq b$ if we can approximate every $f$ belonging to $S$ arbitrarily closely by a linear combination $a_0 y_0 + a_1 y_1 + \cdots + a_k y_k$, that is, technically, if for every $\varepsilon > 0$ we can find constants $a_0, \cdots, a_k$ (with $k$ large enough) such that

\begin{equation}
    \| f - (a_0 y_0 + \cdots + a_k y_k) \| < \varepsilon.
\end{equation}
Hence in the case of completeness every $f$ in $S$ satisfies the so-called **Parseval’s equality**

\[
\sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x)f(x)^2 \, dx.
\]
THEOREM 1

Completeness

Let \( y_0, y_1, \ldots \) be a complete orthonormal set on \( a \leq x \leq b \) in a set of functions \( S \). Then if a function \( f \) belongs to \( S \) and is orthogonal to every \( y_m \), it must have norm zero. In particular, if \( f \) is continuous, then \( f \) must be identically zero.
EXAMPLE 4 Fourier Series

The orthonormal set in Example 1 is complete in the set of continuous functions on \(-\pi \leq x \leq \pi\). Verify directly that \(f(x) \equiv 0\) is the only continuous function orthogonal to all the functions of that set.

Solution. Let \(f\) be any continuous function. By the orthogonality (we can omit \(\sqrt{2/\pi}\) and \(\sqrt{\pi}\)),

\[
\int_{-\pi}^{\pi} 1 \cdot f(x) \, dx = 0, \quad \int_{-\pi}^{\pi} f(x) \cos mx \, dx = 0, \quad \int_{-\pi}^{\pi} f(x) \sin mx \, dx = 0.
\]

Hence \(a_m = 0\) and \(b_m = 0\) in (6) for all \(m\), so that (3) reduces to \(f(x) \equiv 0\).
The power series method gives solutions of linear ODEs

\[ y'' + p(x)y' + q(x)y = 0 \]

with variable coefficients \( p \) and \( q \) in the form of a power series (with any center \( x_0 \), e.g., \( x_0 = 0 \))

\[ y(x) = \sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \]

Such a solution is obtained by substituting (2) and its derivatives into (1). This gives a recurrence formula for the coefficients. You may program this formula (or even obtain and graph the whole solution) on your CAS.
If \( p \) and \( q \) are analytic at \( x_0 \) (that is, representable by a power series in powers of \( x - x_0 \) with positive radius of convergence; Sec. 5.2), then (1) has solutions of this form (2). The same holds if \( \tilde{h}, \tilde{p}, \tilde{q} \) in

\[
\tilde{h}(x)y'' + \tilde{p}(x)y' + \tilde{q}(x)y = 0
\]

are analytic at \( x_0 \) and \( \tilde{h}(x_0) \neq 0 \), so that we can divide by \( \tilde{h} \) and obtain the standard form (1). Legendre's equation is solved by the power series method in Sec. 5.3.
The **Frobenius method** (Sec. 5.4) extends the power series method to ODEs

\[ y'' + \frac{a(x)}{x - x_0} y' + \frac{b(x)}{(x - x_0)^2} y = 0 \]  

(3)

whose coefficients are **singular** (i.e., not analytic) at \( x_0 \), but are “not too bad,” namely, such that \( a \) and \( b \) are analytic at \( x_0 \). Then (3) has at least one solution of the form

\[ y(x) = (x - x_0)^r \sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0(x - x_0)^r + a_1(x - x_0)^{r+1} + \cdots \]  

(4)
where \( r \) can be any real (or even complex) number and is determined by substituting (4) into (3) from the \textbf{indicial equation} (Sec. 5.4), along with the coefficients of (4). A second linearly independent solution of (3) may be of a similar form (with different \( r \) and \( a_m \)'s) or may involve a logarithmic term. \textbf{Bessel's equation} is solved by the Frobenius method in Secs. 5.5 and 5.6.
“Special functions” is a common name for higher functions, as opposed to the usual functions of calculus. Most of them arise either as nonelementary integrals [see (24)–(44) in App. 3.1] or as solutions of (1) or (3). They get a name and notation and are included in the usual CASs if they are important in application or in theory. Of this kind, and particularly useful to the engineer and physicist, are Legendre’s equation and polynomials $P_0, P_1, \cdots$ (Sec. 5.3), Gauss’s hypergeometric equation and functions $F(a, b, c; x)$ (Sec. 5.4), and Bessel’s equation and functions $J_\nu$ and $Y_\nu$ (Secs. 5.5, 5.6).
Modeling involving ODEs usually leads to initial value problems (Chaps. 1–3) or **boundary value problems**. Many of the latter can be written in the form of **Sturm–Liouville problems** (Sec. 5.7). These are **eigenvalue problems** involving a parameter $\lambda$ that is often related to frequencies, energies, or other physical quantities. Solutions of Sturm–Liouville problems, called **eigenfunctions**, have many general properties in common, notably the highly important **orthogonality** (Sec. 5.7), which is useful in **eigenfunction expansions** (Sec. 5.8) in terms of cosine and sine (**“Fourier series”**, the topic of Chap. 11), Legendre polynomials, Bessel functions (Sec. 5.8), and other eigenfunctions.