NONLINEAR ROBUST STATE FEEDBACK CONTROL OF AN OPEN-CHANNEL HYDRAULIC SYSTEM

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Abstract

A nonlinear robust state feedback controller is presented for the minimization of water level deviation in presence of disturbances in a one-reach open channel controlled by an underflow gate at the upstream end. To apply nonlinear robust control to this distributed parameter system, the original nonlinear partial differential equations (the Saint-Venant equations) were adopted without using neither linearization nor discretization. We have defined an infinite-dimensional Hamilton-Jacobi equation and modified an algorithm proposed for solving open loop constrained minimax problem to the infinite-dimensional case. Finally, a receding horizon control is proposed to derive a state feedback control law.

1 Introduction

Over the past several decades, there has been an increasing interest on regulation of irrigation channels based on modern control theory [10, 7, 3, 9]. But there are few of the previous researches that considered both the nonlinear and distributed parameter nature of open-channel hydraulic systems [2]. In irrigation system management, the problem of dealing with unknown variation of water withdrawals/inflows into/from lateral channels is encountered by the engineers. These withdrawals or inflows can be viewed as disturbance inputs. In this paper, this control problem was investigated by using nonlinear robust optimal control theory. The robust optimal control problem was formulated as a two-person differential game in which the disturbances attempt to maximize a performance index, while the control attempts to minimize the same index. In this paper, we formulate a robust problem based on an infinite-dimensional model which was derived directly from the original nonlinear partial differential equations (the Saint-Venant equations) without using neither linearization nor discretization. The disturbance attenuation problem for a nonlinear non-distributed system (i.e. the system under consideration is governed by ordinary differential equations or difference equations) can be resolved by Hamilton-Jacobi-Isaac (HJI) partial differential equation. A robust state feedback control can be computed off-line by several numerical methods [8, 4]. Unfortunately, to our knowledge there is no such approach for a nonlinear distributed parameter system.

This paper consists of three main contributions: first, we defined an infinite-dimensional Hamilton-Jacobi equation. This equation allows us to generalize some well-known results on HJI equation; According to this idea, we secondly modified a minimax control algorithm to cope with our distributed parameter problem. Finally, we applied a receding horizon control strategy to obtain a state feedback control law.

2 Modelling and optimal control of an open-channel hydraulic system

Throughout this example, channel flow is assumed to be one-dimensional and unsteady. This kind of dynamics is governed by the Saint-Venant equations:

\[
\begin{align*}
\frac{\partial h}{\partial t} + \frac{\partial Q}{\partial x} &= 0 \\
\frac{\partial Q}{\partial t} + \frac{\partial (ghQ)}{\partial x} + gA_h \frac{\partial h}{\partial x} - gA(I - J) &= 0
\end{align*}
\]  

(1)

where: \(z(x, t)\) = flow depth, \(Q(x, t)\) = discharge, \(h(x, t)\) = water level, \(A(x, t)\) = cross sectional flow area, \(A_h(x, t)\) = hydraulic cross sectional area, \(h(x, t)\) = water level, \(Q(x, t)\) = discharge, \(I\) = inflow, \(J\) = outflow.

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area, \((m^2)\); \(I\) = canal slope, \((\frac{m}{s})\); \(g\) = acceleration of gravity, \((\frac{m}{s^2})\); \(x\) = distance, \((m)\); \(t\) = time, \((s)\); and \(J\) = friction slope \((\frac{m}{s})\) which can be evaluated using the Manning equation [6]. This problem is subject to the following two initial conditions:

\[Q(x, 0) = \phi_1(x) \quad \text{and} \quad z(x, 0) = \phi_2(x)\]

We assume that \(\phi_1(x)\) and \(\phi_2(x)\) are all known.

As shown in Figure 1, the canal is bounded by an adjustable underflow gate at the upstream end and a spillway at the downstream end, respectively. The discharge of the underflow gate can be derived from the energy equation:

\[Q(0, t) = K_u \sqrt{2g(z_{us} - z_{ds})} = h_1(u(t), z(0, t)) \]

where \(K_u\) represents the product of the discharge coefficient and the width of the gate; the gate opening \(u\) is a function of time; \(z_{us}\) and \(z_{ds}\) are the upstream and downstream water level at the gate, respectively.

In our case, \(z_{us}\) is the water depth of the reservoir \(z_{ur}\), and \(z_{ds}\) is specified as another boundary value \(z(0, t)\). At the downstream end, we assume that the height of spillway is \(h_s\) and the spillway equation is also obtained from an energy equation:

\[Q(L, t) = K(z(L, t) - h_s) \sqrt{2g(z(L, t) - h_s)} = h_2(z(L, t))\]

Therefore, the system (1) is subject to the following boundary conditions:

\[
\begin{cases}
Q(0, t) = h_1(u(t), z(0, t)) & \forall t \in [0, T] \\
Q(L, t) = h_2(z(L, t))
\end{cases}
\]  

\[Q(0, t) = h_1(u(t), z(0, t)) \quad \forall t \in [0, T] \\
Q(L, t) = h_2(z(L, t))
\]  

Our goal is to find a control law in order to minimize the variation of water level along the channel and input discharge of the gate subject to the worst case disturbance, due to unknown inflows or withdrawals. This objective is reflected by the following cost function:

\[
\mathcal{J}_0 = \int_0^T \left[ \int_0^L \frac{p}{2} \left[ z(x, T) - z_0 \right]^2 dx + \int_0^T \frac{1}{2} \left[ u(t) \right]^2 dt \right] + \int_0^T \int_0^L \mathcal{L}(x, t, w(x, t)) dx dt
\]

where \(z_0\) is the reference water depth, \(p > 0\) is a weighting coefficient and \(r > 0\) is the disturbance attenuation coefficient.

### 3 General Problem formulation

In this section, we consider how to derive an open-loop robust control law from a general point of view. For that purpose, we consider the following general partial differential equation:

\[
\begin{cases}
\frac{\partial \xi}{\partial t} = \frac{\partial f}{\partial \xi}(x, t) + g(\xi, w) \\
\xi(x, 0) = \left[ \phi_1(x), \phi_2(x) \right]^T \\
L_{s1}(u(t), \xi(0, t)) = 0 \\
L_{s2}(\xi(L, t)) = 0
\end{cases}
\]

where \(\xi(x, t)\) are the distributed parameter states which depend on the space coordinate \(x\) and the time \(t\), \(u(t)\) is the control input and \(w(x, t)\) is the distributed disturbance. We also denote \(\xi_0(t) = \xi(0, t), \xi_L(t) = \xi(L, t)\) and \(\xi_T(x) = \xi(x, T)\). Our goal is to find some optimal \(u(t)\) and \(w(x, t)\), solutions of a min-max two-player differential game defined by

\[
\mathcal{J}_0 = \int_0^L \phi(\xi_T(x)) dx + \int_0^T \psi(\xi_0(t), \xi_L(t), u(t)) dt + \int_0^T \int_0^L \mathcal{L}(x, t, w(x, t)) dx dt
\]

where \(t \in [0, T], x \in [0, L], T\) and \(L\) are specified. \(\phi(\xi_T)\) represents the terminal performance, \(\psi(u)\) is the performance required for the control input, and \(\mathcal{L}(\xi, w)\) is the desired performance which depend on both the system states and the disturbance.

### 3.1 The Lagrange multiplier method

The Lagrange multiplier method is used together with calculus of variation to deal with the cost functional and all additional constraints (boundary conditions and system dynamics) of the problem. Accordingly, we obtain a new cost functional defined as follows:

\[
\mathcal{J} = \int_0^T \int_0^L \left[ \mathcal{L}(x) + \lambda^T \left( \frac{\partial f}{\partial \xi}(x, t) + g(\xi, w) \right) \right] dx dt + \int_0^T \phi(\xi_T(x)) dx + \int_0^T \psi(\xi_0(t), \xi_L(t), u(t)) dt
\]

where \(\lambda\) are the Lagrange multipliers and \(\Psi(\xi_0, \xi_L, \xi_T, \xi_T)\) is \(\psi(u) + \gamma [L_{s1}(u, \xi_0), L_{s2}(\xi_L)]^T\), where \(\gamma = \left[ \gamma_1, \gamma_2 \right]\) are the Lagrange multipliers of the boundary conditions \([L_{b1}(\xi_0, u, \xi_L), L_{b2}(\xi_L)]\). Introducing Green’s formula into the double integral term of (6), we get

\[
\mathcal{J}(\xi, u, w, \lambda, \gamma) = \int_0^T \left[ \mathcal{L}(x) + \lambda^T f(\xi(x)) \right] dx dt + \int_0^T \int_0^L \left[ H + \left( \frac{\partial \lambda}{\partial \xi} \right)^T \xi \right] dx dt
\]

where we define the Hamiltonian as

\[
H(\xi, \lambda, u, w) = L(\xi, x) + \lambda^T g(\xi, w) - \left( \frac{\partial \lambda}{\partial \xi} \right)^T f(\xi)
\]

Since \(L\) and \(T\) are specified, equation (6) depends on \(\xi, u, w, \lambda\) and \(\gamma\):

\[
\delta \mathcal{J} = \mathcal{J}(\xi + \delta \xi, u + \delta u, w + \delta w, \lambda + \delta \lambda, \gamma + \delta \gamma)
\]

\[
\mathcal{J}(\xi, u, w, \lambda, \gamma) = \int_0^T \left[ \mathcal{L}(x) + \lambda^T f(\xi(x)) \right] dx dt + \int_0^T \int_0^L \left[ H + \left( \frac{\partial \lambda}{\partial \xi} \right)^T \xi \right] dx dt
\]

\[
+ \int_0^T \int_0^L \left[ \psi(\xi_0(t), \xi_L(t), u(t)) \right] dt + \int_0^T \phi(\xi_T(x)) dx + \int_0^T \int_0^L \mathcal{L}(x, t, w(x, t)) dx dt
\]

\[
+ \int_0^T \int_0^L \left[ \frac{\partial H}{\partial u} \delta u \right] dx dt + \int_0^T \int_0^L \left[ \frac{\partial H}{\partial \xi} \delta \xi \right] dx dt + \int_0^T \int_0^L \left[ \frac{\partial H}{\partial \xi_0} \delta \xi_0 \right] dx dt + \int_0^T \int_0^L \left[ \frac{\partial H}{\partial \xi_L} \delta \xi_L \right] dx dt
\]
3.2 Infinite-dimensional Hamilton-Jacobi equation

The adjoint system

\[ \partial J_0^* / \partial t = - \min_{u(t)} \max_{w(x,t)} J_0^* (\xi(t), u(t), w) \]

Expanding \( J_0^* (\xi(t + dt)) \) in Taylor series around the point \( (\xi(t)) \), we obtain

\[ \partial J_0^* / \partial t = - \min_{u(t + dt)} \max_{w(x,t + dt)} J_0^* (\xi(t), u(t), w) \]

The above equation may be called "infinite-dimensional Hamilton-Jacobi equation", since this equation answers the same fundamental questions and will play almost the same role as Hamilton-Jacobi equations for finite-dimensional systems. Its solution allows us to find a control law that satisfies the optimal performance that we have chosen.

Equation (15) can be rewritten as an explicit function of the adjoint states and Hamiltonian from the following computation. First, by computing

\[ dJ = J (\xi(t + dt)) - J (\xi(t)) \]

\[ = - \int_0^L \partial J^*_0 (\xi, t) \frac{\partial \xi}{\partial t} \, dx \]

\[ + \int_0^L \mathcal{L} (\xi(t), w(t)) \, dx \]

and comparing (15) with (16), we obtain

\[ \int_0^L (\partial J^*_0 / \partial \xi) \frac{\partial \xi}{\partial t} \, dx + \Psi (\xi_0, \xi_t, u) \, dt \] (17)

We can thus conclude the evolution of the optimal performance index

\[ \frac{\partial J_0^*}{\partial t} = - \min_{u(t)} \max_{w(x,t)} \{ J_0^* (\xi(t), \xi_t, u) \}
\]

\[ + \int_0^L [H + (\partial J^*_0 / \partial \xi) \frac{\partial \xi}{\partial t} \, dx + \lambda^T f (\xi)] \, dt \] (18)

This result represents a sufficient condition for optimality and allows us to solve it numerically by using the states of the adjoint system. We define the functional \( H \):

\[ H (\xi, u, \lambda) = \Psi (\xi_0, \xi_t, u)
\]

\[ + \int_0^L [H + (\partial J^*_0 / \partial \xi) \frac{\partial \xi}{\partial t} \, dx + \lambda^T f (\xi)] \, dt \] (19)

which is of great interest in the formulation of the numerical algorithm.

3.3 The adjoint-state systems of the robust control problem

In this study, the robust problem can be related to the two following problems respectively: The first is obtained for a given open-loop control \( u(t) \in U [0, T] \),

\[ \max_{u \in W [0, T]} J_0 \] (20)

and the second is obtained for a given disturbance \( w(t) \in W [0, T] \),

\[ \min_{w \in W [0, T]} J_0 \] (21)
Let us consider the Hamiltonians related to these two problems.

$$H_{\text{max}}(\xi, \nu, w) = -\mathcal{L} + \nu^T g(\xi, w) - \left(\frac{\partial g}{\partial w}\right)^T f(\xi)$$

$$H_{\text{min}}(\xi, \lambda, u) = \mathcal{L} + \lambda^T g(\xi, w) - \left(\frac{\partial g}{\partial w}\right)^T f(\xi)$$

(22)

in which $\nu = [\nu_1, \nu_2]^T$ and $\lambda = [\lambda_1, \lambda_2]^T$ are the Lagrange multipliers. According to (10), the solution $w^*$ must satisfy the Saint-Venant equations and their associated adjoint system.

$$\frac{\partial \mathcal{L}}{\partial w^*} = -\left(\frac{\partial g}{\partial w}\right)^T f(\xi) + g(\xi, w^*)$$

(23)

Furthermore, $H_{\text{max}}(\xi^*, \nu^*, w^*) \leq H_{\text{min}}(\xi^*, \nu^*, v)$ for all $v \in W[0, T]$ and $t \in [0, T]$. For the same reason, we have another adjoint system for computing $w^*$.

$$\frac{\partial \mathcal{L}}{\partial w^*} = -\left(\frac{\partial g}{\partial w}\right)^T f(\xi) + g(\xi, w)$$

(24)

And $H_{\text{min}}(\xi^*, \lambda^*, w^*) \leq H_{\text{min}}(\xi^*, \lambda^*, v)$ for all $v \in U[0, T]$ and $t \in [0, T]$, where

$$H_{\text{max}} = -\Psi(\xi_0, \xi_L, u) + \int_0^L \left[H_{\text{max}} + \left(\frac{\partial g}{\partial v}\right)^T f(\xi)\right] dx$$

$$+ [\nu^T f(\xi)]_0^L$$

$$H_{\text{min}} = \Psi(\xi_0, \xi_L, u) + \int_0^L \left[H_{\text{min}} + \left(\frac{\partial g}{\partial w}\right)^T f(\xi)\right] dx$$

$$+ [\lambda^T f(\xi)]_0^L$$

(25)

### 4 Numerical method and simulation

In order to obtain the open loop robust optimal control input $u^*(t)$, there are two essential problems which have to be solved. First, we have to deal with a two-point boundary-value problem: that means the solutions $Q(x, t)$ and $z(x, t)$ of Saint-Venant equations (1), and the solutions $\lambda_1(x, t), \lambda_2(x, t), \nu_1(x, t)$ and $\nu_2(x, t)$ of the adjoint systems (23) and (24) have to be computed for $0 \leq x \leq L$ and $0 \leq t \leq T$. Since these three set of partial differential equations cannot be solved analytically, one naturally thinks of using numerical integration.

The Preissmann scheme, a commonly-used finite difference method developed by hydraulic engineers, was introduced in order to numerically solve the above-defined partial differential equations. Secondly, we have to deal with the min-max problem: in this study the robust problem is formulated as a two-person differential game. We have modified an algorithm which has been proposed by [1] for solving min-max problems of nonlinear non distributed systems. Here we replace the classical Hamiltonians by $H$ in equation (25), since the optimal performance index $J^*_F$ should satisfy the infinite-dimensional Hamilton-Jacobi equations (15) and (18).

Init: Choose small $\epsilon_u, \epsilon_f$ and $d_u^*, d_w^*, d_0^*, d_0^- > 0$ and a first guess $\alpha^0 \geq 0$, $\beta^0 \geq 0$.

S(1): Guess an initial control sequence $u^0(t) \in U$ and $w^0(t) \in W$.

S(2): Compute the system equations (1) forward from $t = 0$ to $t = T$ with the specified initial conditions. Record the solution $\xi_i$.

S(3): Determine the adjoint state $\lambda_i$ and $\nu_i$ by backward computation of the adjoint-state system from $t = T$ to $t = 0$ using $\xi_i$ obtained by S(2) and the final time conditions.

S(4): Using the following formula, compute $u^k, w^k$ for all $t \in [0, T]$.

$$u^k = \arg \min_{u \in U} [H_{\text{min}}^k(\xi^k, u, \lambda^k, \nu^k) + \alpha^{k-1}\|u - u^{k-1}\|^2]$$

$$w^k = \arg \min_{v \in W} [H_{\text{max}}^k(\xi^k, w, \lambda^k) + \beta^{k-1}\|w - w^{k-1}\|^2]$$

where $\xi^k$ is solution of (1) with $u = u^k, w = w^k, \alpha^{k-1}\|u - u^{k-1}\|^2$ and $\beta^{k-1}\|w - w^{k-1}\|^2$ are penalty terms that are introduced to stabilize the successive iterations.

S(5): Compute the following logical variables

$$C_u = [J(u^k, w^{k-1}) > J(u^{k-1}, w^{k-1}) - \epsilon]\{x\}$$

$$C_w = [J(u^{k-1}, w^k) < J(u^{k-1}, w^{k-1}) + \epsilon]$$

$$C_v = [||u^k - u^{k-1}|| > \epsilon_u] \cup [||w^k - w^{k-1}|| > \epsilon_v]$$

S(6): If $C_u$ let $\alpha^{k-1} = \alpha^{k-1} + d_u^+$, if $C_w$ let $\beta^{k-1} = \beta^{k-1} - d_0^-$. 

S(7): If $C_u \cap C_w \cap C_v$ then return to S(4).

S(8): $\alpha^k = \alpha^{k-1} - d_u^-$, $\beta^k = \beta^{k-1} - d_0^-$. 

S(9): If $||\left[\begin{array}{c} u^k \\ u^{k-1} \\ w^k \\ w^{k-1} \end{array}\right] - \left[\begin{array}{c} u^{k-1} \\ u^{k-1} \\ w^{k-1} \\ w^{k-1} \end{array}\right]||^2 \leq \epsilon_u$ or $k \geq k_{\text{max}}$ then stop, else let $k = k + 1$ and return to S(2).

This algorithm [5] searches for a control that corresponds to trajectories satisfying the maximum principle at each time $t \in [0, T]$.

### 4.1 Simulation results for open-loop min-max problem

The simulation program was written in MATLAB and Fortran. The here-proposed nonlinear programming algorithm for solving the Saint Venant equations and their adjoint systems is based on the Levenberg-Marquardt method.

The numerical simulations were performed by using the following parameters: We considered a rectangular channel with the following parameters (main canal of the
"Canal de la Bourne" irrigation system in the south-east of France: length $L = 4 \text{ km}$, width $b = 10 \text{ m}$, bed slope $I = 0.25/1000$, Manning friction coefficient $M = 0.4$, initial gate opening $0.5$, width of the gate $b_0 = b - 0.5 \text{ m}$, min-max gate opening $0.001 - 1.0 \text{ m}$, the gate coefficient $K = 0.7$, water level of reservoir $z_{am} = 1.0 \text{ m}$, height of spillway at the downstream end $h_s = 0.35 \text{ m}$. The simulation parameters are: spatial step $dx = 0.1 \text{ km}$, time sample period $dt = 14 \text{ min}$, Preissmann’s coefficient $\theta = 0.66$, initial water level $\phi_2 = 0.5 \text{ m}$, the initial discharge have to satisfy the relation of stable state $I = J_c$, $\phi_1 = 1.1684 \frac{b^2}{\phi}$ and the boundary condition at $t = 0$, weighting coefficient $p = 1$, initial gate opening $u = 0.5 \text{ m}$, and the disturbance is limited in $W \in [-3(\phi_1/L), 3(\phi_1/L)] \frac{m^3}{s}$. Note that the relaxation procedure terminates conditions are $\epsilon_d = 0.0001$ and $\epsilon_u = 0.0001$. Finally, the control profiles and the behaviors of water level are given on Figure 2.

5 Robust state feedback control

The robust control design proposed in this paper is clearly an open-loop control approach, since the optimal control is associated to a specified set of initial conditions. In order to develop a closed-loop control law, the solution of infinite-dimensional Hamilton-Jacobi equation (18) depending on both $\xi$, $x$ and $t$ is theoretically required. This is most of the time difficult in the finite-dimensional case, and almost impossible in the infinite-dimensional case. The receding horizon control approach is an optimal-control-based method for the design of stabilizing feedback control laws. The attractiveness of this method is we just have to determine an optimal open-loop control, for a given initial state, and we do not have to solve HJI equations like (18).

In the classical receding horizon control approach [11, 12], the open-loop optimal control is solved at each sampling time over the control horizon. Then only the optimal control input at the initial time is applied. The same procedure is repeated at every sampling time to yield a stabilizing feedback control. This strategy can be roughly summarized and illustrated by the following procedure and Figure (4):

1. Use the previously-defined algorithm to obtain the open-loop optimal control law $u^*(\xi(x, 0), 0, T)$ and the worst disturbance $w^*(\xi(x, 0), 0, T)$ and apply it at time $t = 0$.
2. Estimate $\hat{\xi}(x, t + \Delta t)$ at time $t_{es}$, where $dt_e$ is the estimation time.
3. Replace initial condition $\xi(x, 0)$ by the estimation state $\hat{\xi}(x, t + \Delta t)$ and repeat the same algorithm to get $u^*(\hat{\xi}(x, t + \Delta t), t + \Delta t, T)$ and $w^*(\hat{\xi}(x, t + \Delta t), t + \Delta t, T)$ at time $t_{act}$, where $dt_e$ is the computa-

![Figure 2: Robust control behaviors $z_0 = 0.5(m)$, $p = 1$, $J(u^*, w^*) = 0.1710$, $W \in [-3(\phi_1/L), 3(\phi_1/L)]$ and open-loop simulation times $dt_e = 526.477\text{sec}$](image)

![Figure 3: Closed-loop robust control behaviors $z_0 = 0.5(m)$, $p = 1$, $W \in [-3(\phi_1/L), 3(\phi_1/L)]$ and $J(u^*, w^*) = 0.1292$](image)
We should mention here that this application of the receding horizon control strategy is based on the following assumptions: (1) the running time $dt_r + dt_e$ is assumed to be shorter than the sampling time $\Delta t$ for all the $t \in [0, T]$ and all possible state $\xi$. (2) the states are supposed to be completely measurable.

5.1 Application of the receding horizon strategy

We consider the same example as in the open-loop case. Table 1 clearly shows the closed-loop optimal control is less pessimistic than the open-loop case. Since the state feedback implementation is able to detect the effect of the true perturbation through the state measurement.

| worst performance under open-loop robust control ($W \in [-3(\phi_1/L), 3(\phi_1/L)]$) | 0.1710 |
| worst performance under closed-loop robust control ($W \in [-3(\phi_1/L), 3(\phi_1/L)]$) | 0.1292 |

Table 1: comparison between closed-loop performance and open-loop performance

Figure 3 shows the simulation result under the closed-loop robust control. In the simulation example, the open-loop simulation time are $dt_e = 526.477 sec \approx 8.8$ min in average compared to $\Delta T = 14$ min. If the state estimation time are not too long, we believe this computer time is reasonable for real-time implementation.

6 Conclusions

The disturbance attenuation problem of a nonlinear distributed parameter system has been investigated by using a nonlinear closed-loop robust optimal control method based on an infinite-dimensional model. This approach has been successfully applied to the robust control of water level in a one-reach open channel controlled by an underflow gate at the upstream end and subject to inflow or withdrawal disturbances. In future research works, we will consider state-observer design in order to relax the need for full state measurements.

References


